Unbiased Estimation of Ellipses by Bootstrapping

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Abstract—A general method for eliminating the bias of nonlinear estimators using bootstrap is presented. Instead of the traditional mean bias we consider the definition of bias based on the median. The method is applied to the problem of fitting ellipse segments to noisy data. No assumption beyond being independent identically distributed (i.i.d.) is made about the error distribution and experiments with both synthetic and real data prove the effectiveness of the technique.

Index Terms—Implicit models, curve fitting, bootstrap, low-level processing.

1 Introduction—Conic Fitting

Image formation is a perspective projection of the 3D visual environment. Features extracted from a 2D image can be useful only if they preserve some of the geometric properties of the 3D object they correspond to. Collinearity and conicity are such properties, and therefore line and conic segments are widely used as geometric primitives in computer vision.

Let $f(\mathbf{u}, \theta) = 0$ be the implicit model of a geometric primitive in the image. The vector $\mathbf{u} = (x\ y\ 1)^t$ represents the image coordinates, and the vector θ the parameters of the model. For a conic,

$$f(\mathbf{u}, \theta) = \mathbf{u}^t\mathbf{A}\mathbf{u}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \tag{1}$$

and note that the matrix $\mathbf{A}$ is symmetric.

The image points available for estimating the model are $\mathbf{v}_i$, $i = 1, \ldots, n$. The coordinates of these points are often corrupted by noise and have only integer values since the image is defined on a discrete lattice.

Assume that $\mathbf{u}_i$ is the closest point on the curve to the image point $\mathbf{v}_i$, and thus

$$f(\mathbf{u}_i, \theta) = 0, \quad i = 1, \ldots, n. \tag{2}$$

The Euclidean distance between the two points is

$$d_i^2 = (\mathbf{u}_i - \mathbf{v}_i)^t(\mathbf{u}_i - \mathbf{v}_i). \tag{3}$$

To estimate the parameters of the model the following minimization problem has to be solved

$$\min_{\{\mathbf{u}_i\}} \sum_{i=1}^{n} d_i^2 \quad \text{subject to } f(\mathbf{u}_i, \theta) = 0, \quad i = 1, \ldots, n. \tag{4}$$

Using the method of Lagrange multipliers we can define

$$\mathcal{L}(\mathbf{u}_i) = d_i^2 - 2\lambda f(\mathbf{u}_i, \theta), \quad i = 1, \ldots, n \tag{5}$$

and setting

$$\frac{d\mathcal{L}}{d\mathbf{u}_i} = 0 \tag{6}$$

results in

$$\mathbf{u}_i - \mathbf{v}_i = \lambda \nabla f(\mathbf{u}_i, \theta), \tag{7}$$

where $\nabla f(\mathbf{u}_i, \theta)$ is the gradient of $f(\mathbf{u}_i, \theta)$ with respect to $\mathbf{u}_i$.

The quantity $f(\mathbf{v}_i, \theta)$ is known as the algebraic distance of the point $\mathbf{v}_i$ from the curve represented by the implicit model. The first order Taylor expansion of $f(\mathbf{v}_i, \theta)$ around $\mathbf{u}_i$ is

$$f(\mathbf{v}_i, \theta) = f(\mathbf{u}_i, \theta) + (\mathbf{u}_i - \mathbf{v}_i)^t \nabla f(\mathbf{u}_i, \theta) \tag{8}$$

and from (2), (7), and (8) we have

$$f(\mathbf{v}_i, \theta) = \lambda \left\|\nabla f(\mathbf{u}_i, \theta)\right\|^2, \tag{9}$$

where $\left\|\nabla f(\mathbf{u}_i, \theta)\right\|^2$ is the Euclidean norm of the gradient vector. Combining (3), (7), and (9) we obtain the expression

$$d_i^2 = \frac{f^2(\mathbf{v}_i, \theta)}{\left\|\nabla f(\mathbf{u}_i, \theta)\right\|^2} \tag{10}$$

Note that (10) already includes a linear approximation for the algebraic distance. An additional approximation, however, is also required since the norm of the gradient is computed in the unknown point $\mathbf{u}_i$ on the curve. To make the computation possible, instead of $\mathbf{u}_i$ the corresponding point in the image $\mathbf{v}_i$ must be used. Thus, the minimization problem to be solved is

$$\min_{\theta} \sum_{i=1}^{n} \frac{f^2(\mathbf{v}_i, \theta)}{\left\|\nabla f(\mathbf{v}_i, \theta)\right\|^2} \tag{11}$$

In line fitting the denominator is a constant and the solution can be found as an eigenvalue problem ([10], sec. 11.7.2). Recently, Kanatani [14], and Kanazawa and Kanatani [15] have shown that a similar approach, using eigenmatrices, can be used for conics as well. For a conic, the expression (11) is nonlinear in the parameters, and to find the solution iterative methods are required. An extensive discussion of the minimization, involving the use of a second order approximation for the algebraic distance is given in ([10], sec. 11.10).

The conics of interest in image analysis are the ellipses, but most often only a part of the ellipse can be recovered from the data (edge points). Instability of the fit for short noisy conic segments is a serious practical problem [19]. Another important and related issue is the bias of the fit.

The algebraic distance of a point from an ellipse is proportional to the ratio between two Euclidean distances. The distance from ellipse center to a corrupted point is divided by the distance measured along the same direction from the center to its correspondent on the ellipse [1]. The influence of a corrupted image point on the estimated parameters thus depends on the position of that point relative to the ellipse. For example, at the high curvature regions of the ellipse the points on the curve are farther from the center and the influence of the noisy image points is down weighted. The obtained fit will be biased toward smaller ellipses. When the ellipse is estimated from relatively small segments the magnitude of the bias can be significant. Weighting of the algebraic distance with the gradient magnitude, as in (11), does not eliminate the bias; Kanatani [14], and Kanazawa and Kanatani [15] provide a theoretical proof, while Rosin [20] shows the dependence of the distance measure on the relative position of a point.

The difficulty of reliably estimating ellipses yielded a large number of methods. See [21] for a review. However, the problem of bias elimination has not been considered until recently. Porrill [19] proposed an extended Kalman filter formulation incorporating second order information about the implicit model. The Kalman filter assumes Gaussian noise which for integer point coordinates may not hold. The initial values of the filter and the variance of the noise must be provided. The method was also used in [6].

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Kanatani [14], and Kanazawa and Kanatani [15] proposed an iterative renormalization method for bias elimination. The weights of a least squares minimization procedure are updated based on the smallest eigenvalue of the current conic matrix (1) estimate. Werman and Geyzel [23] proposed a resampling technique for unbiased estimation of the conic parameters. The average values of determinants derived from ordered five-tuples of image points are used to compute the parameters. Unbiased estimation of the fifth conic parameter requires that the nature of the noise is known, i.e., that it can be .assumed to be Gaussian. Werman and Geyzel considered only asymptotic unbiasedness and thus the property may not hold for practical sample sizes. A simple empirical iterative procedure for unbiased estimation of circular arcs is described in [12].

Optimum least squares estimation of an unknown 3×3 matrix $\mathbf{A}$ by solving

$$\min \sum_i w_i (\mathbf{x}_i^t \mathbf{A} \mathbf{y}_i)^2 \tag{12}$$

is a generic problem in computer vision. It appears in 3D motion estimation, recovery of 3D structure from uncalibrated cameras, etc. As in the case of conics the estimation is biased, e.g., ([13], sec. 10.2.1), [16], and having a general method for bias reduction is of importance.

In this paper, we propose such a general, numerical method that belongs to the class of resampling techniques. The method is nonparametric, it is based on bootstrap [7], and can be applied to almost any estimator. Since it is nonparametric, no specific (parametric) assumption about the errors is required beyond the usual ones of being centered at zero and independent identically distributed (i.i.d.). The latter, however, can be relaxed [18]. Section 2 starts with a short review of bootstrap and the definition of a less used type of bias, median bias, which is more adequate for computer vision applications. The proposed bias reduction method, Section 3, employs median bias. In Section 4, the new technique is applied to ellipse fitting, both for synthetic and real data.

2 A General Method for Bias Reduction

In this section the main result of the correspondence is presented.

2.1 Bootstrap

Bootstrap is a method for estimating the sampling distribution of a statistic from the available data, and was introduced in statistics by Efron in 1979. For an excellent monograph see [7], for more recent discussions [17], [24]. Bootstrap is yet to be employed for computer vision problems; its only related application is in pattern recognition [11].

The idea of bootstrap is simple. Let the observed sample be $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Under the *errors in variables* model [9] the sample points are corrupted versions of the points $\mathbf{q}_1, \ldots, \mathbf{q}_n$ satisfying a constraint $f(\mathbf{q}, \theta) = 0$, and thus the observations $\mathbf{v}_i$ are modeled by

$$\mathbf{v}_i = \mathbf{q}_i + \epsilon_i \text{ subject to } f(\mathbf{q}_i\ \theta) = 0 \quad i = 1, \ldots, n; \tag{13}$$

where $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. error terms with distribution F. Whenever the noise does not belong to a known parametric family, the distribution F is unknown. In such a case the underlying model (13) of the data is called semiparametric.

Let the empirical distribution of the observed residuals $\hat{\epsilon}_1, \ldots, \hat{\epsilon}_n$ be $\hat{F}$. The estimator of θ is defined by a statistic $T(\hat{F})$ and the observed value of the statistic is $\hat{\theta}$. To obtain the sampling distribution of $\hat{\theta}$ knowledge of F would be required. The plug-in bootstrap principle ([7], ch. 4) substitutes F by $\hat{F}$ and the distribution of $\hat{\theta}$ is simulated by generating N bootstrap samples of size n from $\hat{F}$. Sampling from $\hat{F}$ is the same as sampling with replacement from the set $\hat{\epsilon}_1, \ldots, \hat{\epsilon}_n$. The value of the statistic is calculated for each bootstrap sample, $\hat{\theta}_1^*, \ldots, \hat{\theta}_N^*$, and the bootstrap estimator of the sampling distribution of $\hat{\theta}$ is the empirical distribution of $\hat{\theta}_1^*, \ldots, \hat{\theta}_N^*$. The statistical characteristics of $\hat{\theta}$ are derived from this empirical distribution.

The above described technique is known as bootstrapping residuals [7], and it is the method used in this paper. The details of the implementation are discussed in Section 3.1.

2.2 Median Bias

For a given θ, the bias of the estimator is defined as the difference between the center of the distribution $T(\hat{F})$ and θ. The definition of center can be either the mean or the median. If the distribution of $T(\hat{F})$ given θ is skewed or it has long tails, the median is a better definition of center than the mean. The problem of estimating the ellipse parameters with a least squares estimator belongs to this category. An estimator of θ is called median unbiased if the median of $T(\hat{F})$ given θ is θ,

$$Med_\theta [T(\hat{F})] = \theta. \tag{14}$$

Suppose that $T(\hat{F})$ is a biased estimator of θ under the semiparametric model (13). To reduce the median bias of $T(\hat{F})$ a new estimator is derived through a procedure called target estimation [2], [3]. The median function $g(\theta)$ is defined as

$$g(\theta) = Med_\theta [T(\hat{F})]. \tag{15}$$

This is a function which for each value of θ provides the value of the median of the corresponding sampling distribution of $T(\hat{F})$. Therefore, the target estimator $\hat{\hat{\theta}}$ of θ is

$$\hat{\hat{\theta}} = g^{-1}(\hat{\theta}), \tag{16}$$

given that $g^{-1}(\theta)$ exists. The target estimator is median unbiased if $g(\theta)$ is a strictly monotonic function of θ. The proof is by simple manipulations

$$\mathrm{Med}_\theta[\hat{\hat{\theta}}] = \mathrm{Med}_\theta[g^{-1}(\hat{\theta})] = g^{-1}(\mathrm{Med}_\theta[\hat{\theta}]) = g^{-1}(g(\theta)) = \theta. \tag{17}$$

The necessary and sufficient condition for (17) to hold is the interchangeability of $g^{-1}(\theta)$ and the median operator. The condition of monotonicity of $g(\theta)$ is a stronger one, but it has the advantage that is easier to check.

To derive a target estimator for the mean bias case, the median function must be replaced by the mean function

$$h(\theta) = E_\theta[T(\hat{F})], \tag{18}$$

but the obtained target estimator may not be unbiased. A sufficient condition for mean unbiasedness is that $h(\theta)$ is a linear function, in which case a proof similar to (17) holds. Most of the time, however, the target estimator has less bias than the original estimator. The bias reduction algorithm described in the next section is valid for either definition of the bias.

2.3 Bias Reduction Algorithm

To obtain the reduced bias target estimator $\hat{\hat{\theta}}$ of θ the inverse of the median function has to be computed (16). The analytical expression of $g(\theta)$ is in general not known, and will be estimated through bootstrapping. For the sake of clarity we present the algorithm for a one-dimensional parameter θ, and then indicate how to apply it in the general, multidimensional case.

Assume that an interval (θ_L, θ_U) can be found that contains the solution $\hat{\hat{\theta}}$. This assumption is common for root finding techniques, and it is relatively easy to satisfy by taking a large enough interval. If the direction of the bias is not known, the interval can be centered on $\hat{\theta}$. In this interval m points $(\theta_1, \ldots, \theta_m)$ are selected, equally spaced or at random.

The model (13) is simulated by bootstrapping the residuals N times at each θ_j, $j = 1, \ldots, m$, and the values $\hat{\theta}^*_{j,1}, \ldots, \hat{\theta}^*_{j,N}$ of the estimator are obtained. The initial estimate $g_0(\theta_j)$ of the median function at point θ_j is then the median of $\hat{\theta}^*_{j,k}$, $k = 1, \ldots, N$. The number of simulations N can be relatively small since the smoothing operation required in the following step will use the neighboring values to improve the estimation.

The function $g(\theta)$ is estimated by applying a smoother to the points $[\theta_j, g_0(\theta_j)]$ which produces the points $[\theta_j, g(\theta_j)]$. Employing smoothing splines is a simple and efficient method which can be found in many program packages [4]. The set $[\theta_j, g(\theta_j)]$ gives the numerical approximation of the function $g(\theta)$. This function is almost always monotonic but it is important to check its monotonicity at this stage. The value of the reduced bias target estimator $\hat{\hat{\theta}}$ can now be obtained through interpolation from the set of points $[\theta_j, g(\theta_j)]$ at ordinate $\hat{\theta}$. If $\hat{\hat{\theta}}$ is needed with higher accuracy (16) can be solved by a root finding algorithm [22].

The bias reduction algorithm can summarized as follows.

1) Calculate $\hat{\theta}$ from the data.
2) Define m points in an interval around $\hat{\theta}$. The interval should be chosen large enough to contain the target estimator $\hat{\hat{\theta}}$.
3) Bootstrap the model N times at each point and compute the initial values of the median function.
4) Calculate the median function by smoothing the initial estimates.
5) Obtain $\hat{\hat{\theta}}$ by interpolation.

When the dimension of the parameter is greater than one, i.e. we have a vector θ, the algorithm has a straightforward extension. The m points required in the estimation of the median function are obtained by sampling values of the parameter near $\hat{\theta}$. Any definition of the multivariate median suffices as long as the conditions for (17) hold. For smoothing, multivariate techniques must used, for example MARS [8]. To reduce the amount of computations a trade-off can be made, in which case the median function is estimated component-wise and the smoothing is performed in one-dimension. The accuracy of estimation is however reduced.

The validity of the bias reduction algorithm is limited to those cases where the bootstrap is theoretically applicable. A sufficient condition is $n^{-1/2}$ consistency of the estimator [7], and thus the bias reduction algorithm can be applied in conjunction with both least squares and most robust estimation techniques.

3 Median Bias Elimination in Ellipse Estimation

In Section 1, several methods for unbiased least squares estimation of noisy conic segments were discussed. All these methods considered the case of mean bias and i.i.d. Gaussian noise. The general bias reduction algorithm described in the previous section is now applied to the estimation of ellipses by the least squares estimator (11). The median bias definition is used, and no assumption beside i.i.d. is necessary about the nature of the noise.

3.1 The Procedure

The available data are a sample of $n = 40$ points $\mathbf{v}_i$ additively corrupted from the points $\mathbf{q}_i$ on the ellipse:

$$\mathbf{v}_i = \mathbf{q}_i + \epsilon_i \qquad i = 1, \ldots, n. \tag{19}$$

The two-dimensional error vectors ϵ_i are independent and identically distributed, but their distribution is unknown. Note that due to this noise process the two points, both on the ellipse, $\mathbf{q}_i$ (the uncorrupted version of $\mathbf{v}_i$) and $\mathbf{u}_i$ (the point closest to $\mathbf{v}_i$), are distinct.

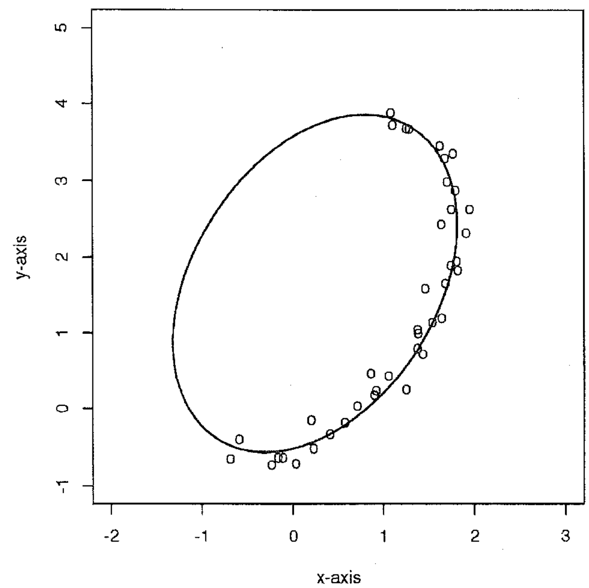

Fig. 1. An example of the noisy data points overlapping the true ellipse.

The ellipse is parametrized by the five-dimensional vector $\theta = (a, b, \alpha, r, c)$, whose components are the minor axis (a), major axis (b), angle of rotation (α), and center of the ellipse (r, c). The angle of rotation is defined relative to the vertical axis and is expressed in radians. The transformation from the general form (1) to this, more intuitive representation, is immediate, e.g., ([10], vol. 1, App. A).

The $n = 40$ points $\mathbf{q}_i$ were chosen equally spaced on a conic arc covering half of an ellipse having the parameters

$$(1.4, 2.32, -0.39, 0.25, 1.65).$$

Both coordinates of a data point $\mathbf{q}_i$ were corrupted with zero-mean Gaussian noise (19) having a standard deviation of 0.16. An example is shown in Fig. 1. Note that the Gaussian nature or the standard deviation of the noise is not needed at any stage of the bias reduction algorithm. From the noisy points $\mathbf{v}_i$ the parameter vector $\hat{\theta}$ was obtained by least squares (11).

A total of $m = 300$ points θ_j, $j = 1, \ldots, m$, were chosen in the five-dimensional space around $\hat{\theta}$. The point $\mathbf{u}_{i,j}$ is defined on the ellipse characterized by θ_j as being the closest to $\mathbf{v}_i$. The residuals are then defined as

$$\hat{\epsilon}_{i,j} = \mathbf{v}_i - \mathbf{u}_{i,j}. \tag{20}$$

From the residuals the bootstrap samples can be generated,

$$\mathbf{v}^*_{i,j} = \mathbf{u}_{i,j} + \hat{\epsilon}^*_{i,j}, \tag{21}$$

where $\hat{\epsilon}^*_{i,j}$ is obtained by sampling with replacement from the set of $\hat{\epsilon}_{i,j}$. For each θ_j, $N = 20$ bootstrap samples of size $n = 40$ were used. From the bootstrap samples the parameter vectors $\hat{\theta}^*_{j,k}$, $k = 1, \ldots, N$ were obtained. The median of these vectors provides the initial value $[\theta_j, g_o(\theta_j)]$. The median function was processed component-wise.

In Fig. 2, the five components of the median function are shown with $m = 300$ points in each plot. Should the estimator $\hat{\theta}$ be unbiased, the points should spread around the zero intercept, unit slope line. The estimated median function, however, is significantly different. Note that the function is monotonic as required by the algorithm.

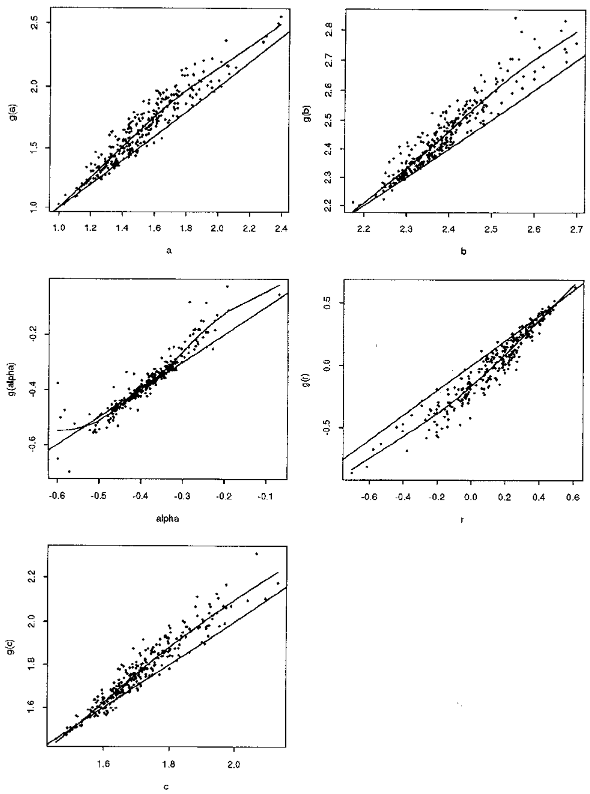

Fig. 2. The five components of the median function. In each plot, the median function is the smooth curve, the straight line corresponds to an unbiased estimator.

3.2 Experimental Results

To analyze the statistical properties the above procedure was repeated 300 times. Fig. 3 shows the comparison between the sampling distributions of the five parameter estimates before and after median bias correction. The boxplots of the five components of the least squares estimated parameter vectors are shown at the left and the bias corrected estimates at the right of each pair. The horizontal line represents the true value of the parameters, the dot the median of the sampling distribution, and the box is bounded by the 25th and 75th percentiles. The skewness and the median bias of the distribution of the least squares estimators is significant for all parameters except α.

As Fig. 3 shows, after application of the bias reduction algorithm, the median bias has disappeared and the variance of the estimates has also been reduced. The reduction in the variance depends on the shape of the median bias function. The effect of bias reduction on the estimated ellipses can be observed in Fig. 4 Fig. 4a shows a random subset of 40 ellipses fitted by least squares and Fig. 4b shows the corresponding corrected ellipses. The darker line represents the original ellipse.

In order to evaluate the effectiveness of the nonparametric bootstrap a second simulation was performed in which the same procedure was repeated except that instead of generating the errors from the empirical distribution of the residuals the true error distribution was used. (The technique is known as parametric bootstrap.) The performance was indistinguishable from that of the nonparametric bootstrap. We can conclude that for a sample size of $n = 40$ the empirical distribution of the errors is as good as the true distribution for estimating the sampling distribution of the estimator.

The obtained results also compare favorably with those published in the literature, and obtained under stricter assumptions. Kanatani [14] provides eccentricity-area scatter diagrams to characterize the behavior of the bias corrected estimator, Porrill [19] presents results as in Fig. 4.

To investigate the behavior of the bias reduction algorithm for real data an image similar to those used in [14], [15], [19] was generated. Only a few edge pixels from the rim of a *coffee mug* were used to estimate the whole ellipse (Fig. 5a). The results obtained with least squares estimation and corrected for bias are shown in (Fig. 5b), and the latter is close to the correct result in spite of not using any assumption about the noise (which is severely correlated due to quantization).

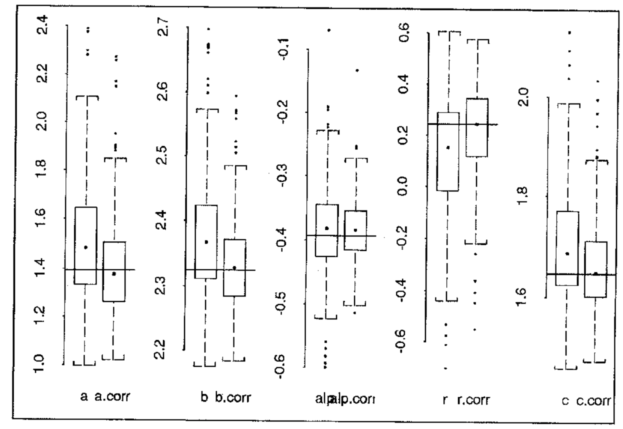

Fig. 3. Boxplots of the sampling distribution of the least squares estimator of the five ellipse parameters before (left) and after bias correction. The true value of each parameter is shown by a horizontal line. The median bias is the difference between the line and the dot.

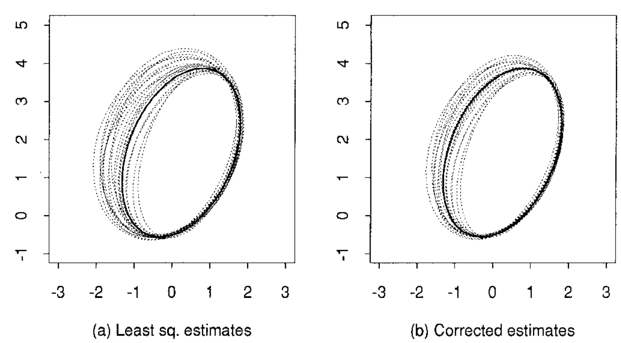

Fig. 4. Examples of the estimated ellipses before and after bias correction. The darker line corresponds to the true ellipse.

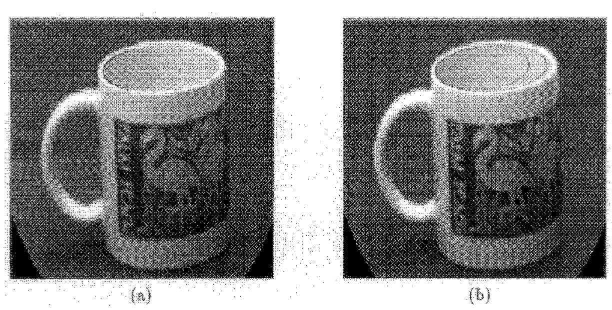

Fig. 5. (a) The *coffee mug* image with the data points used in the ellipse estimation marked as black; (b) The least squares estimate of the ellipse (inner curve) and the bias corrected one.

4 Conclusion

We have described a general methodology for bias and median bias reduction of nonlinear estimators. Such estimators arise frequently in computer vision problems and reduction of the bias of the estimates is of importance. The method is not based on a priori (parametric) assumptions about the error distribution and its performance compares favorably with the case in which these distributions are available. Application of bootstrap to computer vision problems is not restricted to bias reduction. The sampling distribution of the output, derived from the available single input, can also be used to define the confidence in the obtained result [5]. Any resampling method is computer intensive but easily parallelizable, in which case a significant performance improvement is obtained at the expense of relative small overhead.

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