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COURSE 11

NEW OUTLOOKS AND OLD DREAMS IN QUANTUM ANTIFERROMAGNETISM

P. Chandra

NEC Research Institute,
4 Independence Way, Princeton, NJ 08540, USA

P. Coleman

Serin Physics Laboratory, Rutgers University,
P.O. Box 849, Piscataway, NJ 08854, USA

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Contents

1. Introductory remarks

1.1. A brief historical interlude

Quantum antiferromagnets are an "economy" class of strongly interacting electron fluid; despite their deceptive simplicity in the absence of charge motion, they can display a rich phase behavior characteristic of complex many-body systems. Indeed, since its theoretical conception [1,2] in the 1930s, quantum antiferromagnetism has often served to sharpen and to develop our conceptual understanding of strongly correlated matter; the Bethe Ansatz [3], broken symmetry [4], semiclassical "large-S" quantum mechanics [5,6], and frustration [7] are but a short list of broad contributions that this field has made to date. In the strongly correlated electron problem the delicate interplay between interactions and fluctuations often leads to novel order, and here we shall explore this theme within a pure spin context.

The possible destabilization of long-range antiferromagnetic order is an old notion, particularly as the staggered magnetization is not a conserved order parameter. In the days preceding neutron scattering it was widely believed that Bethe's exact solution of the spin-$\frac{1}{2}$ antiferromagnetic chain [3] could be philosophically extended to higher dimensions, implying the absence of any ordered antiferromagnetic ground state. In 1951 neutron diffraction experiments demonstrated the presence of Néel order in manganese oxide [8]; in the following year Anderson [5] and Kubo [6] independently developed a semiclassical theory to explain these results. Making a strong analogy with the quantum harmonic oscillator, Anderson stressed the importance of the zero-point energy, which had been neglected in earlier treatments. Within a leading-order expansion in $(ZS)^{-1}$, he showed that the reduction in the sublattice magnetization is

$$\delta M \sim \int \mathrm{d}^d q \left[n(\omega_q) + \frac{1}{2S} \right] \frac{1}{\omega_{\vec{q}}} \qquad (\omega_{\vec{q}} \sim cq,\ q \to 0), \tag{1.1}$$

where S is the spin; eq. (1.1) is analogous to the "equipartition" expressions $\langle x^2 \rangle_{\mathrm{CM}} \sim 1/\omega^2$ and $\langle x^2 \rangle_{\mathrm{QM}} \sim \hbar/(2\omega)$ for the classical and quantum harmonic oscillator, respectively. For a d-dimensional hypercubic lattice $\omega_{\vec{q}} \sim q$, so Néel order is possible at $T = 0$ for $d \geqslant 2$. Similarly eq. (1.1) indicates that in one-dimension

eading-order quantum corrections result in an infrared divergence, consistent with Bethe's result. With the rapid growth of neutron sources, many experiments on wo- and three-dimensional isotropic antiferromagnets have been performed; the neasured magnetization shows excellent agreement with the zero-point spin reluction calculated in this fashion [9,10].

Despite the widespread success of spin-wave theory, there remain a number of nagnetic insulators that defy description within this approach. In 1973 Anderon noted the distinct *absence* of observed two-dimensional $S = \frac{1}{2}$ Heisenberg ntiferromagnets, and remarked that in low-dimensional, low-spin frustrated systems quantum fluctuations might indeed destroy the Néel order [11]. The situation hanged dramatically in 1987 with the discovery of $LaCuO_4$, a 2D spin-$\frac{1}{2}$ Heisenerg antiferromagnet. This material has the unusual property, when doped, of going directly from a semimetallic to a superconducting state. Anderson immediately roposed that here large fluctuations might destroy the sublattice magnetization, eading to a "spin-liquid" state; the addition of charge fluctuations would then lead o a novel form of superconductivity [12].

Anderson's suggestion has revived interest in an old question in magnetism: an a *gapless* phase be stabilized in the absence of long-range antiferromagnetic rder? The most recent attack on this problem is characterized by a "quantum luids" approach to spin systems, thus liberating magnetism from a semiclassical 'solid" framework (fig. 1). For the sake of completeness we note that the analogy etween antiferromagnetism and superfluidity is not new, but has been previously tressed only for long-wavelength modes [14–16]. The microscopic "two-fluids" erspective on antiferromagnetism brings several new methods to this problem e.g. gauge theories, large-N expansions, homotopy theory), tools successful in ther correlated quantum systems; quantum antiferromagnets thus acquire a new tatus as "economy" strongly interacting electron fluids.

The real motivation for this fresh approach to quantum antiferromagnetism omes from experiment; several real systems are characterized by short-range pin correlations coexisting with low-energy magnetic excitations, thereby sugesting new forms of spin order. Of course, the most enticing materials exhibting this behavior are the cuprate superconductors, where charge doping of a Néel ntiferromagnet [17] leads to novel superconducting and normal-state behavior. lere, inelastic neutron [18] and nuclear-magnetic resonance experiments [19] ndicate low-energy spin fluctuations, in seeming contradiction with the meaured short spin correlation length [20]. Of course, charge doping introduces *both* tatic and dynamic "frustration", and is thus more complicated than the pure spin nodels discussed here [21]; however, we shall see that even the behavior of a rongly fluctuating *spin* fluid in the absence of charge fluctuations is not straightorward.

There is now a growing list of anomalous antiferromagnetic materials whose

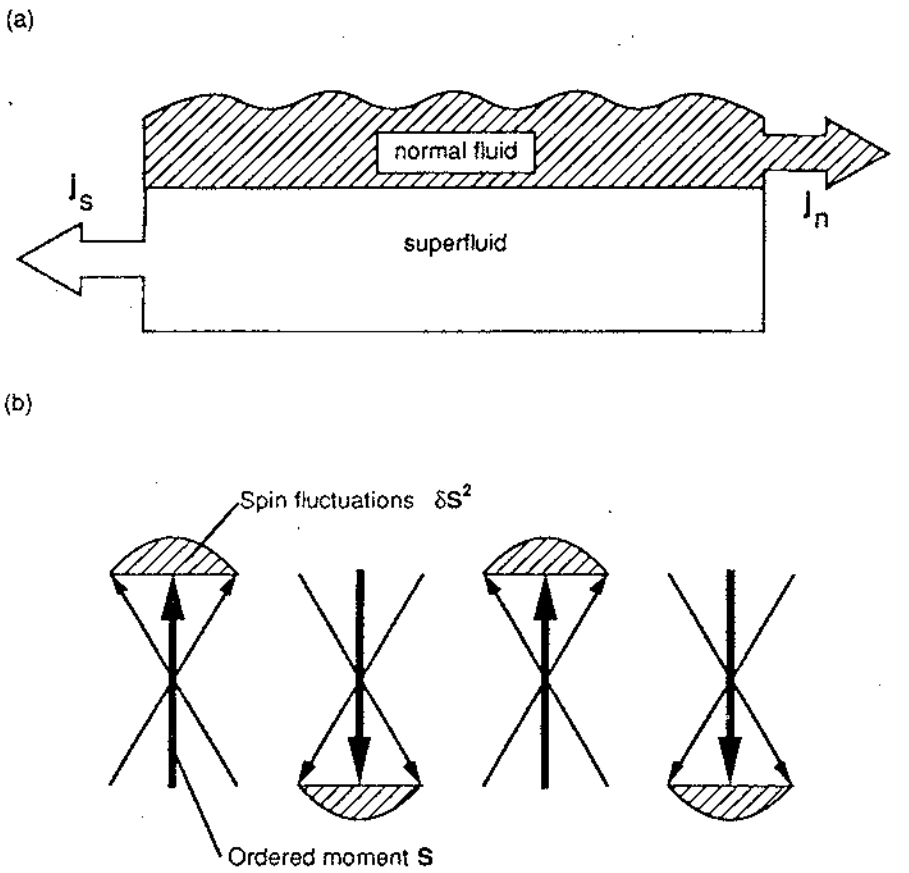

Fig. 1. A schematic representation of the contrast between (a) the fluids and (b) the "rigid" semiclassical approach to antiferromagnetism. Reprinted from Ritchey [13].

novel features demand a new approach to magnetism. Traditionally a magnetic insulator has been viewed as a rigid array of weakly interacting local moments, a picture that is only valid in the limit of small spin fluctuations. Table 1 shows a number of magnetic "mysteries", systems with very small (if finite!) ordering temperatures despite the presence of strong antiferromagnetic correlations. Many of these materials have structures that impose geometrical constraints on the ground-state spin configuration, thereby strongly enhancing the spin fluctuations. Roughly speaking, the Curie–Weiss temperature θ_{CW} is an indication of the expected ordering temperature in the absence of such "frustration"; Ramirez et al. [22] have

Table I

A few mysterious magnetic materials. Here T^* refers to a magnetic ordering temperature.

Material	Lattice	$-\theta_{CW}$ [K]	T^* [K]	Novel features
$LiNiO_2$	2D □ ($S=\frac{1}{2}$)	210	65	Strong Ising anisotropy, evidence for zero-point entropy and glassiness
VCl_2	2D △ ($S=\frac{3}{2}$)	437	36	Evidence for vortex unbinding transition
$Gd_3Ga_4O_{12}$	3D garnet ($S=\frac{7}{2}$)	2.3	< 0.03	Absence of order in 3D AFM
$SrCr_8Ga_4O_{19}$	2D ✡ ($S=\frac{3}{2}$)	492	3.5	Observed features of both spin glass and ordered AFM

characterized these enigmatic magnets by a large "frustration function"

$$f_R = \frac{\theta_{CW}}{T^*} \tag{1.2}$$

where T^* is the measured ordering temperature. Indications of "moment-free" phases with possible textures in these unconventional magnets suggest the need for a new fluids approach that can handle their partially ordered behavior.

1.2. *Low-dimensionality and frustration: a recipe for strong fluctuations*

How can spin fluctuations in Heisenberg magnets be enhanced? Could such strong fluctuations promote "moment-free" spin ground states with new forms of magnetic order? Dimensionality plays a crucial role in this discussion; as d is decreased, the phase space available for long-wavelength fluctuations is enhanced. At the lower critical dimension, long-wavelength fluctuations completely suppress classical antiferromagnetism at all values of spin S. This feature can be seen clearly from the spin-wave expression for the moment reduction eq. (1.1), which has contributions from both thermal fluctuations,

$$\delta M \sim T \int \frac{d^d q}{q^2}, \tag{1.3}$$

and quantum zero-point motion,

$$\delta M \sim \int \frac{d^d q}{q}, \tag{1.4}$$

that diverge as $d \to 2$ and $d \to 1$, respectively.

The results may also be understood by a simple rescaling of the long-wavelength action; here we assume that the presence of an ordered antiferromagnetic state is determined by spin waves, so that only configurations with neighboring spins differing infinitesimally from each other need consideration. In the simplest bipartite antiferromagnets, the long-wavelength dynamics are determined purely by the gradients of the staggered magnetization in space and time, so that we may approximate the classical Heisenberg Hamiltonian

$$H = J \sum_{(i,j)} \vec{S}_i \vec{S}_j \tag{1.5}$$

by its continuum counterpart; the associated long-wavelength action is

$$S_C = \frac{H}{T} = \frac{1}{2g} \int d^d x \, (\nabla \hat{n})^2, \tag{1.6}$$

where $g = T/(2JS^2 a^{d-2})$ is a coupling constant, a is the lattice spacing, and $\hat{n}$ is a unit vector in the direction of the staggered magnetization. Since the dimension of the coupling constant is $2-d$, a rescaling of length-scales

$$x \to x' = \frac{x}{b} \qquad (b = 1 + d\Lambda) \tag{1.7}$$

leads to a renormalized coupling constant

$$g \to g^* = g b^{d-2} = g\,[1 + (d-2)\, d \ln \Lambda] \tag{1.8}$$

or

$$\frac{dg}{d \ln \Lambda} = (d-2)g, \tag{1.9}$$

which indicates that $d_C = 2$ is the lower-critical dimension for classical antiferromagnets. In a similar fashion, quantum antiferromagnets are described by gradients of the order parameter in both space and time,

$$S_Q = \frac{1}{2\tilde{g}} \int_0^\beta d\tau \int d^d x \left[\frac{1}{c^2} (\partial_\tau \hat{n})^2 + (\nabla \hat{n})^2 \right], \tag{1.10}$$

which is a $d+1$-dimensional field theory, with a coupling constant of dimension $2-(d+1) = 1-d$; the lower critical dimension of a quantum antiferromagnet is thus one ($d_Q = 1$), explaining the absence of an ordered moment in quantum spin chains.

In these lectures we focus on two-dimensional quantum antiferromagnets. Be-

cause they are in the vicinity of $d_Q = 1$ their zero-temperature long-wavelength fluctuations are strong; they are thus good "hosts" for the development of exotic spin ordering. Furthermore, because $d_C = 2$, short-wavelength thermal fluctuations can induce discrete lattice symmetry-breaking unaccompanied by a conventional magnetic transition, due to the *finite* nature of the spin correlation length. Thus short-wavelength fluctuations assume a particular importance in two-dimensional spin systems, a topic we shall discuss in sections 3 and 4.

The role of fluctuations is further enhanced in *frustrated* spin systems, where the energy of each spin bond cannot be minimized simultaneously. A degenerate ground-state manifold often results; *anisotropic* thermal and quantum fluctuations can then select new ordered phases. This "order from disorder", first discussed by Villain [23], is a short-wavelength phenomenon and thus does not depend on the presence of long-range antiferromagnetic order [23–25]. Frustration also modifies the nature of the underlying order parameter, changing the homotopy group associated with the allowed defects (see fig. 2) [26]. The order parameter for the unfrustrated 2D square Heisenberg model, for example, is a vector; spin vortices are thus topologically unstable, since they can always relax by twisting out of the plane into the third dimension [26]. By contrast, in the frustrated 2D triangular antiferromagnet the order parameter is described by three orthogonal vectors; π twists of this order parameter form "Z_2 vortices" that are *topologically stable* [27]. Indeed, a generalized Kosterlitz–Thouless topological transition, associated with the binding of these defects, has been proposed by Kawamura and Miyashita for the triangular case [27], and we shall return to this question in section 6.

In these lectures we hope to introduce the reader to the new "economy" approach to strongly correlated electron systems, focusing primarily on the case of frustrated two-dimensional antiferromagnets. We begin with a general description of the antiferromagnet as a quantum fluid, emphasizing the conceptual framework for this picture. Spin currents, the Marshall sign theorem, and the importance of the Onsager cavity method for strongly fluctuating spin systems are discussed before a more technical gauge-invariant treatment of quantum antiferromagnetism is presented (section 3). Next we introduce the notion of "order from disorder", demonstrating with a simple example how the interplay of fluctuations and interactions can promote the selection of new spin ground states in the *absence* of conventional antiferromagnetic order (section 4). Fluctuation-suppression of the sublattice magnetization leads to the possibility of exotic states at the "brink of disorder"; the "*quantum zoo*" of candidates (e.g. spin insulators, Laughlin spin liquids, spin nematics) is reviewed in section 5. In section 6 we turn to a particularly challenging and controversial example of a strongly fluctuating spin system: the two-dimensional Heisenberg kagomé antiferromagnet, which has its experimental realization in two magnetoplumbite materials. Here we present a case for why the classical system is *not* a conventional magnet; the possibility of a (non-Abelian)

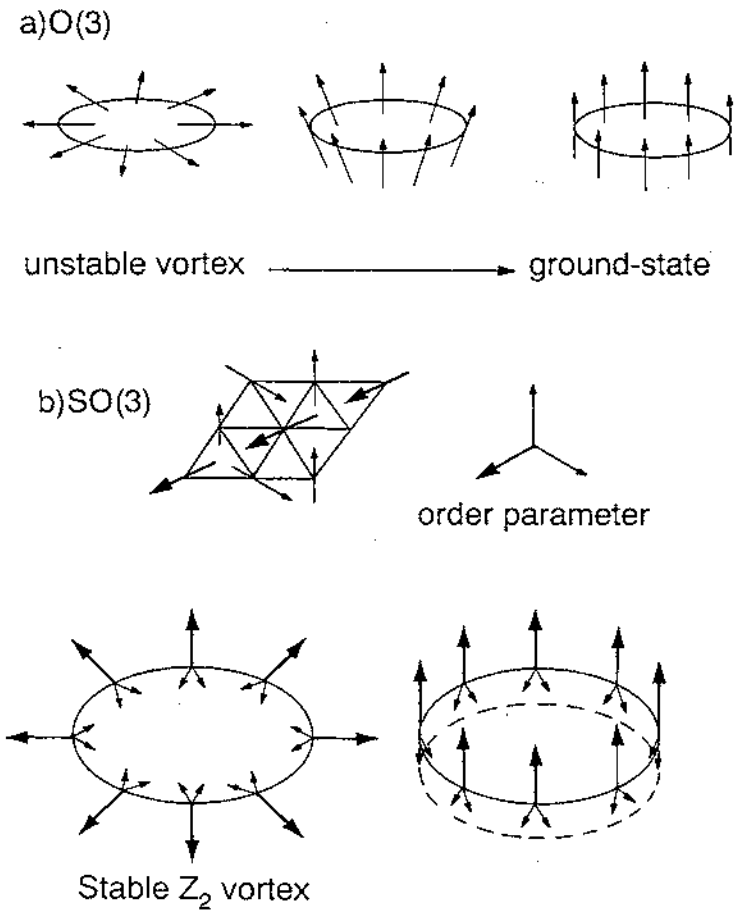

Fig. 2. (a) Unstable versus (b) stable point defects in the square (unfrustrated) and the triangular (frustrated) two-dimensional Heisenberg antiferromagnets, respectively.

defect-mediated glass transition in this problem is discussed, with implications for experiment. Finally, we end (section 7) with some concluding remarks and many open questions.

2. The antiferromagnet as a quantum fluid: concepts

The renewed interest in strongly fluctuating two-dimensional Heisenberg models demands a gauge-invariant approach to antiferromagnetism, one that encompasses

the semiclassical results of spin-wave theory *and* survives the loss of a moment. Initiating a new perspective to the problem, Anderson has proposed a variational resonating valence bond (RVB) approach to Heisenberg spin systems [12], one drawing close analogy between antiferromagnetism and helium-4. Figure 3 shows the stark contrast between the RVB and the Néel states on a triangular lattice; it is not at all obvious that the "fluids" approach can recover the more conventional results. In a seminal paper, Liang, Doucot, and Anderson (LDA) showed that an RVB state with finite-range spin bonds can generate long-range antiferromagnetic order [28]; the "nodeless" nature of the RVB wavefunction yields a two-fluids picture of magnetism. The antiferromagnet is now no longer treated as a rigid magnetic structure, but as a quantum fluid. Before discussing this approach in more detail, let us list a few questions that naturally arise in such an exercise:

(1) What new insights are to be gained from a "fluid" perspective of antiferromagnetism?

(2) What is a spin liquid and does spin flow exist? Is there a spin current?

(3) A "quantum fluid" approach to spin systems suggests many new types of ground states, in analogy with the strongly correlated electron problem. For example, are there spin analogues of insulators, Fermi liquids, superfluids, and Laughlin liquids? How and when are these possibilities realized?

(4) Are there spin textures analogous to those found in helium and in liquid crystals?

These issues have been the focus of much recent research activity, and are certainly far from settled. Here we begin with a discussion of Marshall's sign rule, a key ingredient in the Liang–Doucot–Anderson demonstration [28] that an RVB wavefunction can recover long-range antiferromagnetic order. The resulting form of this wavefunction naturally leads to a "fluid" picture of antiferromagnetism. Next we develop the notion of a spin current, discussing the continuity of spin flow and the connection between broken spin rotation invariance and spin superfluidity. A rotational-invariant treatment of magnetism must contain the essential physics of spin fluctuations, and therefore must extend the concept of the Weiss field to cases where the local moment vanishes. This has been done by Brout and Thomas [29] in the context of disordered Ising magnets; they have applied Onsager's idea [30] of a non-orienting reaction field to spins. We end with a discussion of the Onsager reaction field for strongly fluctuating Heisenberg magnets; its microscopic realization is provided by the Schwinger boson approach, the central topic of the next section.

2.1. *The Marshall sign rule*

A crucial property of the ground-state wavefunction associated with the bipartite antiferromagnet is its symmetry or "nodelessness", a feature suggesting a bosonic

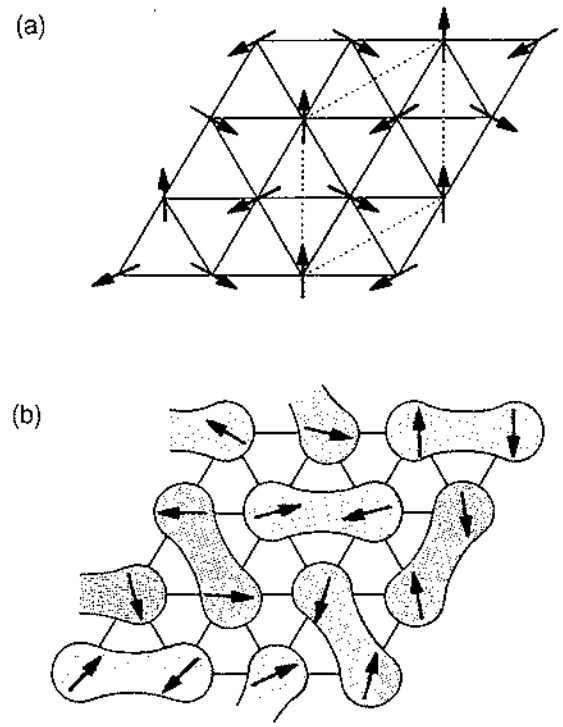

Fig. 3. An illustration of the stark contrast between (a) the Néel and (b) the resonating valence bond (RVB) states on the triangular lattice, where the shaded regions in (b) represent the singlet bonds between the spins. The RVB phase is a linear combination of all such coverings of the lattice. Reprinted from Ritchey [13].

representation and thus the possibility of condensation. Marshall [31] noted that under the basis transformation

$$\left(S^{\perp}(x,y), S^{z}(x,y)\right) \rightarrow \left((-1)^{x+y} S^{\perp}(x,y), S^{z}(x,y)\right), \tag{2.1}$$

which rotates all spins on the A sublattice by π, the Heisenberg Hamiltonian

$$H = J \sum_{(i,j)} \vec{S}_i \cdot \vec{S}_j \tag{2.2}$$

becomes

$$H' = gHg^{\dagger} = J \sum_{(i,j)} (S_i^z S_j^z - \vec{S}_i^{\perp} \cdot \vec{S}_j^{\perp}), \tag{2.3}$$

where

$$\vec{S}_i^{\perp} \cdot \vec{S}_j^{\perp} = \tfrac{1}{2}(\sigma_i^+\sigma_j^- + \sigma_i^-\sigma_j^+) \tag{2.4}$$

in the usual fashion.

Marshall emphasized that all the off-diagonal components of the transformed Hamiltonian are *negative*, so that its ground-state eigenvector is nodeless by Frobenius' theorem. Because this result plays a central role in our discussion, we will outline the argument. Briefly, the transformed Hamiltonian $H' = |i\rangle H'_{ij}\langle j|$ can be written

$$H'_{ij} = \epsilon_i\delta_{ii} - h_{ij} \qquad (h_{ij} = 0,\ i = j), \tag{2.5}$$

where h_{ij} are the off-diagonal elements. Let us consider any normalized state vector

$$|\psi\rangle = \sum_i \psi_i|i\rangle \tag{2.6}$$

and its nodeless counterpart

$$|\tilde{\psi}\rangle = \sum_i |\psi_i|\,|i\rangle. \tag{2.7}$$

Then the expectation value of the energy in the nodeless state $\tilde{E} = \langle\tilde{\psi}|H'|\tilde{\psi}\rangle$ is always less than or equal to the energy in the "nodeful" state, since

$$\begin{aligned} E &= \epsilon_i|\psi_i|^2 - \sum_{i>j} h_{ij}\left\{\psi_i^*\psi_j + \psi_j^*\psi_i\right\} \\ &\geqslant \epsilon_i|\psi_i|^2 - \sum_{i>j} 2h_{ij}|\psi_i|\,|\psi_j| = \tilde{E}. \end{aligned} \tag{2.8}$$

It follows that the ground-state eigenvector in the staggered reference frame is nodeless, with only positive components. Let the basis states $|i\rangle \equiv |\{x_i\}\rangle$ be labelled by the coordinates $\{x_i\}$ of the down spins, then the ground state may be written

$$\Psi' = \sum_{\{x_{i\downarrow}\}} |\psi(x_{i\downarrow})|\,|x_{1\downarrow},\ x_{2\downarrow}, \cdots, x_{N/2\downarrow}\rangle. \tag{2.9}$$

We would now like to transform back to the original spin coordinates. To do so, the wavefunction of each down electron on the A sublattice must be multiplied by $e^{i\pi} = -1$, resulting in a ground-state wavefunction

$$\psi(x_i) = (-1)^{n_{A\downarrow}}\,|\psi(x_i)|, \tag{2.10}$$

where $n_{A\downarrow}$ refers to the number of down spins on the A sublattice. The expression (2.10) is known as the Marshall sign rule; it demonstrates that under a suitable transformation the ground-state wavefunction is always *nodeless*, suggesting the possibility of condensation into a single momentum state, thus providing a cornerstone for the two-fluids approach to magnetism.

2.2. *The Liang–Doucot–Anderson wavefunction*

Liang, Doucot, and Anderson (LDA) noted that the singlet ground state of the bipartite Heisenberg antiferromagnet could be represented as a linear superposition of valence-bond states, corresponding to all singlet bond configurations on the lattice [28]. Respecting Marshall's sign rule, LDA associated a positive amplitude $f(i_\alpha - j_\beta)$ with a singlet pair

$$(i,j) = \frac{1}{\sqrt{2}}(\uparrow_i\downarrow_j - \downarrow_i\uparrow_j) \qquad (i \in A,\ j \in B)$$

between sites i and j on the A and the B sublattices, respectively; the resulting class of trial RVB wavefunctions are

$$\Psi_{\mathrm{RVB}} = \sum_{(i_\alpha, j_\alpha)} \prod_\alpha f(i_\alpha - j_\alpha)(i_\alpha, j_\alpha), \tag{2.11}$$

where the bond-strength distribution $f(l)$ as a function of bond length (l) is a variational parameter (fig. 4). Optimizing this LDA wavefunction (2.11) using a variational Monte Carlo algorithm [32] on large (180×180) two-dimensional lattices, LDA found that the lowest-energy states were associated with *power-law* pairing $f(l) \sim l^{-p}$ for $2 < p < 3$ and a staggered magnetic correlation function $C(\vec{r}) = (-1)^{(x+y)}\langle\vec{S}(\vec{r})\vec{S}(0)\rangle$ with a spin correlation length of order the lattice size [28].

The Liang–Doucot–Anderson results indicate that a linear superposition of *finite-range* singlet bonds can recover long-range antiferromagnetic order; this situation is reminiscent of helium-4, where short-range pairing of the helium atoms leads to a rigid Bose condensate [33]. To make this analogy more transparent we use a bosonic representation for the spins, originally due to Wigner and Schwinger [34], but most recently developed by Arovas and Auerbach [35]. Here, a spin S is represented by a symmetric wavefunction of $n = 2S$ spin-$\frac{1}{2}$ bosons (fig. 5). More formally, this "Schwinger boson" representation takes the form

$$\begin{aligned} &\vec{S}_i = \tfrac{1}{2}b_{i\sigma}^\dagger\vec{\sigma}_{\sigma\sigma'}b_{i\sigma'} \\ &b_{i\sigma}^\dagger b_{i\sigma} = 2S, \end{aligned} \tag{2.12}$$

$$\Psi_{\mathrm{LDA}} = \sum_{\{i_\alpha, j_\alpha\}} \prod_{\substack{i\in A \\ j\in B}} f\,(i_\alpha - j_\alpha)\,(i_\alpha, j_\alpha)$$

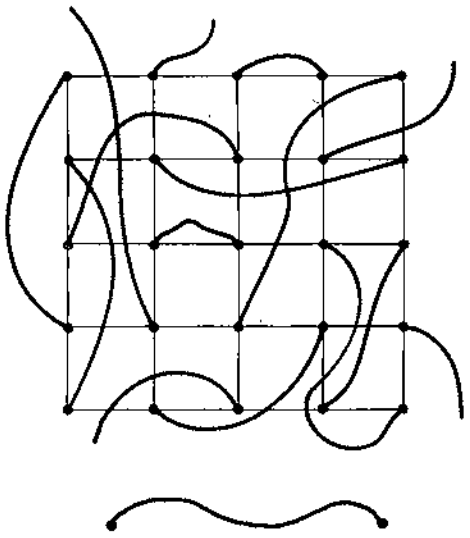

Fig. 4. A pictorial summary of the Liang–Doucot–Anderson trial wavefunctions; here the singlet ground state of the bipartite antiferromagnet is represented as a linear superposition of valence-bond states with positive bond weights.

where $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices; it permits us to regard a lattice of spin S moments as a fluid of paired spin-$\frac{1}{2}$ bosons [36] with constant density per site $\rho = 2S$. From this perspective, a large-S classical magnet is the high-density limit of quantum antiferromagnetism.

Let us now rewrite the LDA wavefunction in a second-quantized form. In the Schwinger representation a singlet bond (i, j) is created by the Bose pairing operator

$$B^\dagger_{ij} = b^\dagger_{i\uparrow} b^\dagger_{j\downarrow} - b^\dagger_{i\downarrow} b^\dagger_{j\uparrow}\,, \tag{2.13}$$

which becomes a *symmetric* triplet spin pair in Marshall's staggered reference frame

$$B^\dagger_{ij} \longrightarrow B'^\dagger_{ij} = g B^\dagger_{ij} g^\dagger = b^\dagger_{i\sigma} b^\dagger_{j\sigma}\,. \tag{2.14}$$

In this language, the *nodeless* LDA wavefunction is then

$$\Psi'_{\mathrm{RVB}} = g\Psi_{\mathrm{RVB}} = \sum_{\{i_\alpha\in A,\, j_\alpha\in B\}} \prod_\alpha f(i_\alpha, j_\alpha) B'^\dagger_{ij}\,|0\rangle. \tag{2.15}$$

We can reexpress eq. (2.15) in a form that clearly shows the nature of the spin pairing; to do so, we introduce a "Gutzwiller" operator P_{2S} ($2S = 1$) that projects out the wavefunction component with $2S$ bosons per site, and then exchange the sum and the product in eq. (2.15), rewriting it as:

$$\Psi'_{\mathrm{RVB}} \propto P_{2S}\left(\sum_{i,j} f(i-j) B'^\dagger_{ij}\right)^{NS} |0\rangle, \tag{2.16}$$

where N is the number of lattice sites. The right side of eq. (2.16) appears as the NSth term in a series expansion of an exponential; it will be selected by the projection operator, and eq. (2.16) can thus be rewritten in Jastrow form

$$\begin{aligned}\Psi'_{\mathrm{RVB}} &\propto P_{2S}\,\exp\left(\sum_{i,j} f(i-j) B'^\dagger_{ij}\right)|0\rangle \\ &= P_{2S}|\Psi_b\rangle, \\ |\Psi_b\rangle &= \exp\left(\sum_{\vec{q}} f_{\vec{q}}\, b^\dagger_{\vec{q}\uparrow} b^\dagger_{-\vec{q}\downarrow}\right)|0\rangle,\end{aligned} \tag{2.17}$$

where the Fourier transformation of

$$f_{\vec{q}} = \frac{1}{2N}\sum_{\vec{r}} f(\vec{r})\, e^{-i\vec{q}\cdot\vec{r}} \tag{2.18}$$

gives the bond strengths.

We define the pair of bosonic amplitudes that satisfy

$$\begin{aligned} u^2_{\vec{q}} - v^2_{\vec{q}} &= 1 \\ \frac{v_{\vec{q}}}{u_{\vec{q}}} &= f_{\vec{q}}, \end{aligned} \tag{2.19}$$

yielding $u_{\vec{q}} = [1 - f^2_{\vec{q}}]^{-1/2}$, $v_{\vec{q}} = f_{\vec{q}}/[1 - f^2_{\vec{q}}]^{1/2}$. The corresponding Bogoliubov quasiparticle spin destruction operator

$$a^\dagger_{\vec{q}\sigma} = u_{\vec{q}} b^\dagger_{\vec{q}\sigma} - v_{\vec{q}} b_{-\vec{q}-\sigma} \tag{2.20}$$

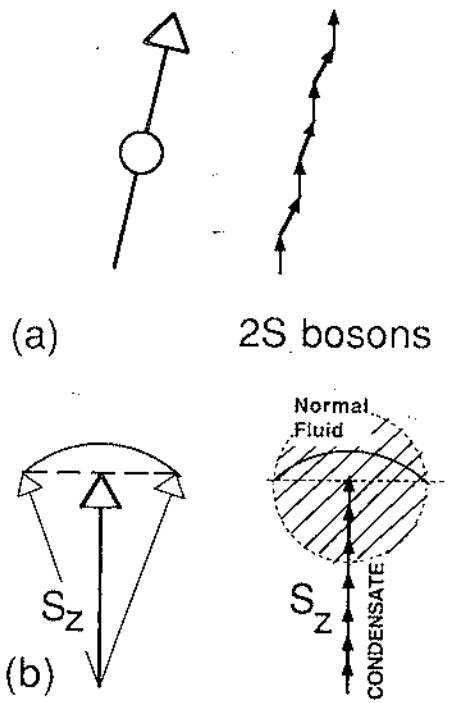

Fig. 5. The "Schwinger boson" approach to magnetism. (a) A pictorial view of how a spin S is built with $2S$ spin-$\frac{1}{2}$ bosons. (b) A schematic view of the two-fluid picture; the normal fluid describes the spin fluctuations while classical magnetism is the condensate.

has the following properties

$$[a_{\vec{q}\sigma}, A^n] = -nA^{n-1} v_{\vec{q}} b_{\vec{q}\sigma},$$
$$a_{\vec{q}\sigma} \exp[A] = -\exp[A]\, v_{\vec{q}} b_{\vec{q}\sigma},$$

$$A \equiv \sum_{\vec{q}} f_{\vec{q}}\, b^\dagger_{\vec{q}\uparrow}\, b^\dagger_{-\vec{q}\downarrow}, \tag{2.21}$$

and hence annihilates the unprojected LDA wavefunction

$$a_{\vec{q}\sigma}\,|\Psi_b\rangle = 0. \tag{2.22}$$

This bosonic representation of the Liang–Doucot–Anderson wavefunction was first discussed by Read and Sachdev [37].

In the thermodynamic limit ($N \to \infty$) $f_{\vec{0}} \to 1$, resulting in an infinite accumulation of particles in the $\vec{Q} = 0$ state ($\langle b^\dagger_{\vec{0}}\rangle \to \sqrt{NS^*}$); the LDA wavefunction (2.17) can be divided into a normal and a "classical" component

$$|\Psi_{\rm RVB}\rangle = P_{2S}|\Psi_{\rm N}\rangle|\Psi_{\rm C}\rangle, \tag{2.23}$$

with

$$|\Psi_{\rm N}\rangle = \exp\left\{\sum_{q\neq 0} f_{\vec{q}}\, b^\dagger_{\vec{q}} b^\dagger_{-\vec{q}}\right\}|0\rangle,$$
$$|\Psi_{\rm C}\rangle = \exp\left\{\sqrt{NS^*}\,[b^\dagger_{0\uparrow} + b^\dagger_{0\downarrow}]\right\}|0\rangle, \tag{2.24}$$

where we note that condensation will occur at the wavevector $\vec{Q} = (\pi, \pi)$ in the untwisted reference frame [38].

The presence of a spin condensate will be very sensitive to the nature of the spin pairing. Within this treatment, the value of the ordered moment is set by the constraint $\langle n\rangle = 2S$. In the high-density limit, where P_{2S} in eq. (2.17) may be ignored, this constraint may be written

$$2S = \frac{1}{N}\sum_{\vec{q}\sigma}\langle\Psi_B|b^\dagger_{\vec{q}\sigma} b_{\vec{q}\sigma}|\Psi_B\rangle$$
$$= \frac{1}{N}\sum_{\vec{q}\sigma} v^2_{\vec{q}}\langle\Psi_B|a_{-\vec{q}\sigma} a^\dagger_{-\vec{q}\sigma}|\Psi_B\rangle = \frac{2}{N}\sum_{\vec{q}}\frac{f^2_{\vec{q}}}{1-f^2_{\vec{q}}}, \tag{2.25}$$

where we have rewritten the Schwinger bosons in terms of quasiparticle operators

$$b_{\vec{q}\sigma} = u_{\vec{q}} a^\dagger_{\vec{q}\sigma} + v_{\vec{q}\sigma} a_{-\vec{q}-\sigma} \tag{2.26}$$

to evaluate the matrix element in eq. (2.25). To ensure proper normalization of the LDA wavefunction, f_0 is determined by the mean-field condition (2.25). The right-hand side of eq. (2.25) is clearly maximized when $f_0 = 1$; if $2S$ exceeds this value, a macroscopic occupation of the $q = 0$ state develops in the thermodynamic limit (i.e. $v_0^2 \sim NS^*$) to satisfy the constraint. The value of the ordered moment, S^*, is determined by

$$S^* = S - \int \frac{{\rm d}^d q}{(2\pi)^d}\frac{f^2_{\vec{q}}}{f^2_0 - f^2_{\vec{q}}}, \tag{2.27}$$

where S^* and the fluctuation integral represent the condensate and normal fluid contributions, respectively.

Equation (2.27) indicates that the development of long-range antiferromagnetic order is sensitive to both dimensionality and to the bond-distribution in momentum space. For example a nearest-neighbor RVB state, first considered by Sutherland [39], is characterized by $f_{\vec{q}}^2 \sim f^2(1-q^2)$ as $f \to 1$; thus the fluctuation integral in eq. (2.27) diverges ($\sim \ln[1/(1-f^2)]$) and any large value of S can be achieved without the development of a $q = 0$ condensate. Following LDA, one can also choose a power-law bond strength $f(l) \sim l^{-p}$ and thus $f_{\vec{q}} - 1 \sim q^{p-2}$; for $d = 2$ the normal fluid integral is bounded for $p < 4$, implying that *long-range* pairing is crucial for the development of a sublattice magnetization in two dimensions. This is in agreement with the numerical results of LDA; they found an ordered state for all $p < 4$, and achieved their lowest energies for $2 < p < 3$. We note that in three dimensions the fluctuation integral associated with a Sutherland dimer state is convergent ($S_c \sim 1.57$) even for $f_{q=0} = 1$, suggesting that for $S > S_c$ the 3D Sutherland RVB wavefunction develops long-range antiferromagnetic order (see table 2).

Table 2

RVB wavefunctions and long-range order.

$f_{\vec{q}}$	d	S_c
$c_x + c_y + c_z$ (short range)	3	1.5(7) ordered $S > S_c$
$c_x + c_y$ (short range)	2	∞ never ordered
$R^{-P}\cos\vec{q}\cdot\vec{R}$ (long range)	2(3)	$P < 4(5)$ ordered for $S = \frac{1}{2}$

From these simple arguments, we see that an ordered antiferromagnet can be regarded as a two-component quantum fluid. We shall now extend the analogy with the quantum fluid further, identifying the condensate with classical Néel antiferromagnetism and the "normal" fluid with fluctuations about this ordered state (see fig. 5). In the classical large-S limit, the "normal" spin fluid is only important in low dimensions, whereas in frustrated, small-S systems "order from disorder" interactions in the normal fluid can play an important role in selecting the low-temperature spin ground state.

2.3. *Spin currents and spin conductivity*

A "fluids" approach to magnetism implies the presence of local spin flow; how can we define a spin current if it cannot be measured directly? More generally,

can we discuss a spin conductivity if there is no analogue of an electric field to induce spin flow? To do so, we must regard the current as a response to a sudden change in boundary conditions. In conventional charged superfluids, for example, where the Landau–Ginzburg free energy is

$$F = \frac{\rho_s}{2}\int d^d x\,(\hbar\vec{\nabla}\phi - q\vec{A})^2 \qquad (\rho_s = |\Psi_0|^2), \tag{2.28}$$

the particle supercurrent is then

$$\vec{j}(x) = -\frac{1}{q}\frac{\delta F}{\delta \vec{A}(x)} = \rho_s[\hbar\vec{\nabla}\phi(x) - q\vec{A}(x)], \tag{2.29}$$

where ρ_s is the superfluid density.

The free-energy functional (2.28) is gauge-invariant under the local transformation

$$\begin{aligned}\phi(x) &\to \phi(x) + \alpha(x),\\ \vec{A}(x) &\to \vec{A}(x) + \left(\frac{\hbar}{e}\right)\nabla\alpha(x),\end{aligned} \tag{2.30}$$

indicating that a change in phase boundary conditions is physically equivalent to a change in the vector potential. We can also impose a *non-periodic* boundary condition $\phi(L) = \phi(0) + \Delta\theta$ ($\Psi(L) = e^{i\Delta\theta}\,\Psi(0)$) on a torus of superfluid with length L; it can be absorbed by a gauge transformation $\alpha(L) = -\Delta\theta$ where the associated vector potential satisfies

$$\Delta\theta = -\frac{e}{\hbar}\int_0^L A_l(\vec{r})\,dr_l, \tag{2.31}$$

and such a change of boundary conditions is physically equivalent to a magnetic flux containing $\Phi = (\hbar/e)\Delta\theta$. This result follows directly from the local gauge invariance associated with local continuity of flow, and thus is not restricted to the case of superfluids.

The boundary conditions can also be changed with time; this is gauge-equivalent to a time-dependent vector potential or an electric field

$$eE_x = -e\frac{\partial A_x}{\partial t} = \hbar\frac{\Delta\dot{\theta}}{L}. \tag{2.32}$$

If this process is suddenly imposed at time $t = 0$, creating an effective field pulse, eq. (2.32) becomes

$$eE(t) = \hbar\frac{\Delta\theta}{L}\delta(t), \tag{2.33}$$

with a resulting particle-current pulse

$$j(\omega) = \tilde{\sigma}(\omega)\hbar\frac{\Delta\theta}{L},$$

where $\tilde{\sigma}(\omega) = \sigma(\omega)/e^2$ is the particle conductivity, and does not depend on the specific particle charge. A supercurrent is signaled by the development of a pole of strength $(q/e)\rho_s$ at zero frequencies in the conductivity function.

This argument relies solely on particle conservation. In neutral fluids, where currents do not couple to the physical vector potential, one can introduce a fictitious pure gauge vector potential as a book-keeping device [33]. The system's response to this fictitious gauge field is physically equivalent to that associated with time-dependent boundary conditions, or to an external potential gradient. The generalization of conductivity to such neutral systems is thus a response to a suddenly imposed change of boundary conditions.

Local continuity of flow is a key feature of fluidity; in a charged superfluid it is associated with a local gauge invariance. There, the invariance of the free energy under the gauge transformation

$$\begin{aligned}&\Psi(x) \to e^{i\theta(x)}\,\Psi,\\ &(\phi, \vec{A}) \to (\phi + \frac{\partial\theta}{\partial t}, \vec{A} + \nabla\theta)\end{aligned} \tag{2.34}$$

establishes the local continuity equation

$$\frac{\delta F[\theta]}{\delta\theta(x)} = \frac{\partial\rho_s(x)}{\partial t} + \vec{\nabla}\cdot\vec{j}(x) = 0,$$

where Ψ is the complex order parameter.

A similar approach can be taken in quantum spin systems. Conventionally we attribute a Heisenberg antiferromagnet with *global* spin conservation associated with spin rotational invariance. However, the motion of spin is a continuous process and there will be associated spin continuity equations relating the divergence of spin currents to the precession of the local moments [38]. Following the neutral superfluid analogy, the long-wavelength action [40,41] for the Heisenberg antiferromagnet can be written [38]

$$I = \frac{JS^2}{2}\int d^dx \sum_i (\nabla_i\hat{n} + \vec{A}_i \times \hat{n})^2, \tag{2.35}$$

where $M = S\hat{n}$ and A is a fictitious curl-free vector potential used for determining the spin currents. In analogy with the conventional superfluid, coherent spin

currents result from the broken SU(2) rotational gauge invariance associated with global spin conservation. These currents are then

$$\vec{j}_i = -\frac{\delta I}{\delta\vec{A}_i} = JS^2\nabla_i\hat{n}\times\hat{n}, \qquad \nabla_i = (\nabla_i + \vec{A}_i\times), \tag{2.36}$$

and correspond to twisted spin configurations.

Let us now generalize this discussion to the case of lattice spin systems described by Heisenberg models. The Heisenberg Hamiltonian must be rewritten in a gauge-invariant form; for electrons on a lattice we have

$$H = \sum_{ij} t_{ij}\psi_i^\dagger \exp\left(i\int_i^j \vec{A}\cdot d\vec{x}\right)\psi_j, \tag{2.37}$$

so that one might expect that analogously for spins on a lattice the Hamiltonian is

$$H = \sum_{ij} J_{ij}\vec{S}_i \exp\left(i\int_i^j \vec{A}\cdot d\vec{x}\right)\vec{S}_j. \tag{2.38}$$

However, spins are vectors, where

$$[S^\alpha]_{ab} = i\epsilon_{a\alpha b}, \tag{2.39}$$

so that

$$i[A_l^\alpha S^\alpha] \equiv -(\vec{A}_l\times), \tag{2.40}$$

and the gauge-invariant form of the Heisenberg Hamiltonian [38] is

$$H[\vec{A}_l] = \tfrac{1}{2}\sum J_{ij}\vec{S}_i \exp\left(-\int_i^j \vec{A}_l\, dx_l\times\right)\vec{S}_j. \tag{2.41}$$

The resulting spin current from k to i is

$$\vec{j}_{k\to i} = -\left.\frac{\delta H}{\delta\vec{A}_{ik}}\right|_{\vec{A}=0} = J_{ik}(\vec{S}_k\times\vec{S}_i). \tag{2.42}$$

In this language the spin continuity equation,

$$\begin{aligned}\frac{d\vec{S}_i}{dt} &= \sum\vec{j}_{k\to i} = J_{ik}(\vec{S}_k\times\vec{S}_i)\\ &= \vec{S}_i\times -\sum J_{ik}\vec{S}_k,\end{aligned} \tag{2.43}$$

is equivalent to the equation for spin precession in the presence of a Weiss magnetic field, the second term on the right-hand side of eq. (2.43).

We can expand the gauge-invariant form of the Heisenberg Hamiltonian (2.41) in powers of the spin vector potential

$$H[\vec{A}_l] = H - \sum_{\vec{x}} \vec{A}_l(\vec{x})\vec{j}_l(\vec{x}) + \tfrac{1}{2}\sum_{\vec{x}} A_l^{\alpha}(\vec{x})\mathcal{N}^l_{\alpha\beta}(\vec{x})A_l^{\beta}(\vec{x}), \tag{2.44}$$

where

$$\vec{j}_l(\vec{x}) = -\frac{\partial H}{\partial \vec{A}_l(\vec{x})} = \tfrac{1}{2}\sum_{\vec{R}} J(\vec{R})R_l\vec{S}(\vec{x} - \tfrac{1}{2}\vec{R}) \times \vec{S}(\vec{x} + \tfrac{1}{2}\vec{R}),$$

$$\mathcal{N}^l_{\alpha\beta}(\vec{x}) = -\frac{\partial^2 H}{\partial A_l^{\alpha} A_l^{\beta}} = \tfrac{1}{2}\sum_{\vec{R}} J(\vec{R})(R_l)^2\,[S^{\alpha}(\vec{x} - \tfrac{1}{2}\vec{R})S^{\beta}(\vec{x} + \tfrac{1}{2}\vec{R}) - \delta^{\alpha\beta}\vec{S}(\vec{x} - \tfrac{1}{2}\vec{R}) \cdot \vec{S}(\vec{x} + \tfrac{1}{2}\vec{R})]. \tag{2.45}$$

The derivative with respect to the spin vector potential yields the spin current

$$\vec{\mathcal{J}}_l = \vec{j}_l - \mathcal{N}^l \cdot \vec{A}_l \qquad ([\mathcal{N}^l]_{\alpha\beta} = \mathcal{N}^l_{\alpha\beta}). \tag{2.46}$$

Analogous to the superconducting case, the first term on the right-hand side (rhs) of eq. (2.46) is the instantaneous "diamagnetic" response to an external twist field, where the second "paramagnetic" contribution corresponds to the adiabatic relaxation of the imposed twist. The linear response to a spin vector potential is

$$\vec{\mathcal{J}}_l(x) = -\sum_{\vec{x}'}\int_{\tau'} \gamma^l(\vec{x},\vec{x}';\tau - \tau')\vec{A}_l(x')$$

$$\gamma^l(\vec{x},\vec{x}';\tau - \tau') = \mathcal{N}^l(\vec{x}')\delta^3(x - x') - \langle \mathrm{T}\, j_l(x)j_l(x')\rangle, \tag{2.47}$$

where we have used the shorthand $x \equiv (\vec{x},\tau)$, $\delta^3(x) \equiv \delta_{\vec{x}}\delta(\tau)$ and suppressed the spin indices; here the spin stiffness is analogous to the London kernel for a superconductor.

Are antiferromagnets truly spin superfluids? In a spin fluid with unbroken rotational invariance, the diamagnetic and paramagnetic response terms in eq. (2.46) exactly cancel at long times and distances, renormalizing the spin-wave stiffness to zero. However, when this spin rotational symmetry is broken, spin-current fluctuations *incompletely* screen the diamagnetic response; the rigidity of the wavefunction leads to a *finite* spin stiffness tensor. Does a finite spin stiffness imply spin superfluidity? In general, no! Superfluid flow depends *not only* on the stiffness of the ground-state wavefunction but also on the *topological stability* of vortices that

permit a macroscopic circulating supercurrent. In a conventional superfluid, persistent currents around a ring are associated with a phase change $2\pi n$; here current loss cannot occur in a continuous fashion, because the phase change around the ring is a topological invariant. Similarly, the development of spin superflow demands the topological stability of spin vortices; in this sense an xy magnet with a U(1) order parameter is a true "spin superfluid". The situation is more complicated for Heisenberg spin systems: for uniaxial vector antiferromagnets (O(3)), vortices can "escape into the third dimension" [26], although it is possible to have *stable* 2π defects for non-collinear antiferromagnets (SO(3)). However here, unlike the U(1) case, a vortex is its own antiparticle and a state with a 4π order parameter twist can distort back into the ground state. In this sense, no macroscopic spin currents are possible: the topology of the order parameter actually suppresses macroscopic spin currents, and despite their *finite* superfluid stiffness Heisenberg magnets are *spin insulators* (see fig. 6). In practice, however, weak xy anisotropy or finite boundaries *stabilize* such defects, and there is also the possibility of a richer class of stable spin textures associated with higher homotopy groups.

2.4. *The Onsager reaction field*

Any treatment of strongly fluctuating antiferromagnetism that aims to recover conventional semiclassical results must extend the concept of a Weiss field to cases where the local moment vanishes. Such a generalization has been studied by Brout and Thomas [29], who have adapted Onsager's reaction field [30] to the spin problem. Specifically, a spin is only sensitive to the "cavity field" it would experience at its own location, and the correlated polarization of its neighboring spins does not play a role in its orientation. Following the treatment of Brout and Thomas [29], the spin's orienting field is the sum of the uniform Weiss and the "reaction" ($\langle\delta\vec{B}_i\rangle$) fields associated with the formation of an empty cavity at site i (fig. 7); the second contribution is determined by removing the spin at site i and computing the resulting field. To compute this feedback effect, we suppose that the"cavity" spin at site i has a small polarization $\langle\delta\vec{S}_i\rangle$ which would have induced a Weiss field $\vec{B}[k] = -J_{ki}\langle\delta\vec{S}_i\rangle$ at neighboring sites j in the *absence* of the cavity; thus with the creation of the cavity these spins at j "feel" a change in their Weiss field given by

$$\delta\vec{B}^{\text{cavity}}[k] = -\vec{B}[k] = J_{ki}\,\delta\vec{S}_i\,, \tag{2.48}$$

where J_{ij} is the coupling constant between sites i and j (fig. 7). Equation (2.48) then induces a change in magnetization at site j

$$\langle\delta S_j\rangle^{\text{cavity}} = \sum_{k\neq i}\chi_{jk}\,\delta\vec{B}^{\text{cavity}}[k] = \chi_{jk}J_{ki}\langle\delta\vec{S}_i\rangle^{\text{cavity}}, \tag{2.49}$$

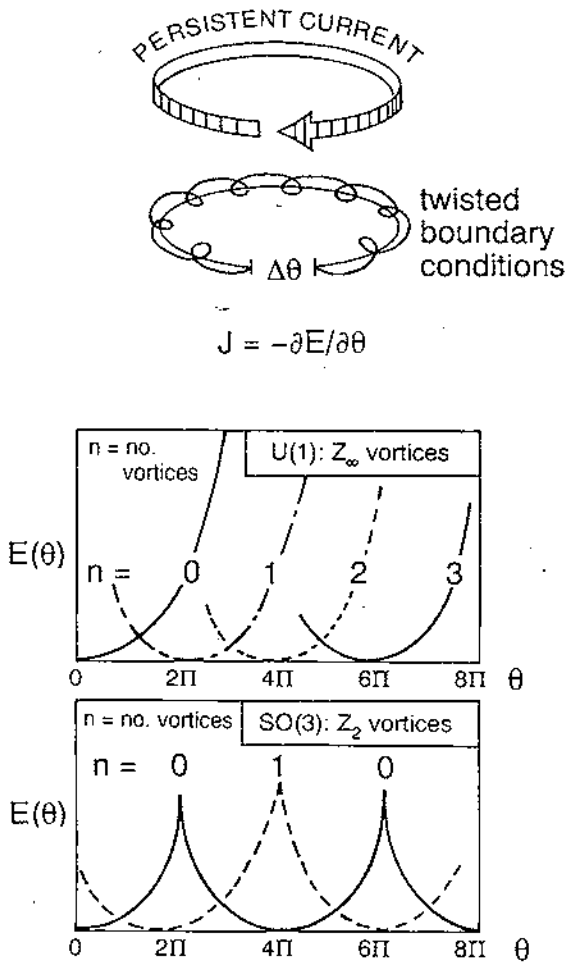

Fig. 6. A pictorial representation of the link between a coherent spin current and twisted boundary conditions. In an xy (U(1)) magnet $J = -\partial E(\theta)/\partial\theta$ is *macroscopic* and thus there are persistent spin currents; this is *not* the case for the SO(3) helimagnet.

which, in turn, produces a Weiss field at the site of the cavity, given by

$$\langle \delta\vec{B}_i^{\text{cavity}} \rangle = -J_{ij}\langle \delta\vec{S}_j \rangle^{\text{cavity}}, \tag{2.50}$$

so that

$$\langle \delta\vec{B}_i \rangle^{\text{cavity}} = -2\mu\langle \delta\vec{S}_i \rangle^{\text{cavity}}, \tag{2.51}$$

where

$$2\mu = \sum_{jk} J_{ij}\chi_{jk}J_{ki} = \int_{\vec{q}} J^2(q)\chi(q). \tag{2.52}$$

Here $\chi(\vec{q})$ is the zero-frequency susceptibility of the antiferromagnet at wavevector $\vec{q}$, and

$$\int_{\vec{q}} = \int_{\text{BZ}} \frac{\mathrm{d}^d q}{(2\pi)^d} \tag{2.53}$$

is the momentum integral over the Brillouin zone. Thus, in the cavity of the surrounding spins, the spin at site i experiences a feedback field $\langle B_i \rangle^{\text{cavity}}$ which is proportional to the spin polarization at that site.

The corresponding energy per site associated with zero-point and thermal spin fluctuations is then

$$E_{\text{ZP}} = -\frac{1}{2N}\sum \langle \vec{B}_i \cdot \vec{S}_i \rangle = \mu S(S+1). \tag{2.54}$$

Equation (2.54) permits us to relate the Onsager reaction field constant μ with the spin fluctuations in the system

$$\mu = \frac{1}{2S(S+1)}\int_{\vec{q}} J(\vec{q})\,\langle \vec{S}_{-\vec{q}} \cdot \vec{S}_{\vec{q}} \rangle_\mu\,. \tag{2.55}$$

In order to satisfy the fluctuation dissipation theorem

$$\langle \vec{S}_{-\vec{q}} \cdot \vec{S}_{\vec{q}} \rangle_\mu = \int \frac{\mathrm{d}\omega}{\pi}\,[\tfrac{1}{2} + n(\omega)]\,\mathrm{Im}\,\chi(\vec{q},\omega) \tag{2.56}$$

the Onsager reaction field constant must be adjusted *self-consistently* [29] to satisfy *both* equations (2.52) and (2.55), the dynamic and static expressions for μ. Such self-consistency implies

$$\int_{\vec{q}} J^2(q)\chi(q) = \int_{\vec{q}} J(\vec{q}) \int \frac{\mathrm{d}\omega}{\pi}\,[\tfrac{1}{2} + n(\omega)]\,\mathrm{Im}\,\chi(\vec{q},\omega), \tag{2.57}$$

where the susceptibility is computed with the Onsager feedback field $\vec{B}_i = -2\mu\vec{S}_i$ replacing the standard Weiss field. These equations are sufficient in their own right for the study of strongly fluctuating magnets, and in principle any method can be used to compute the dynamical susceptibility in the presence of the Onsager field.

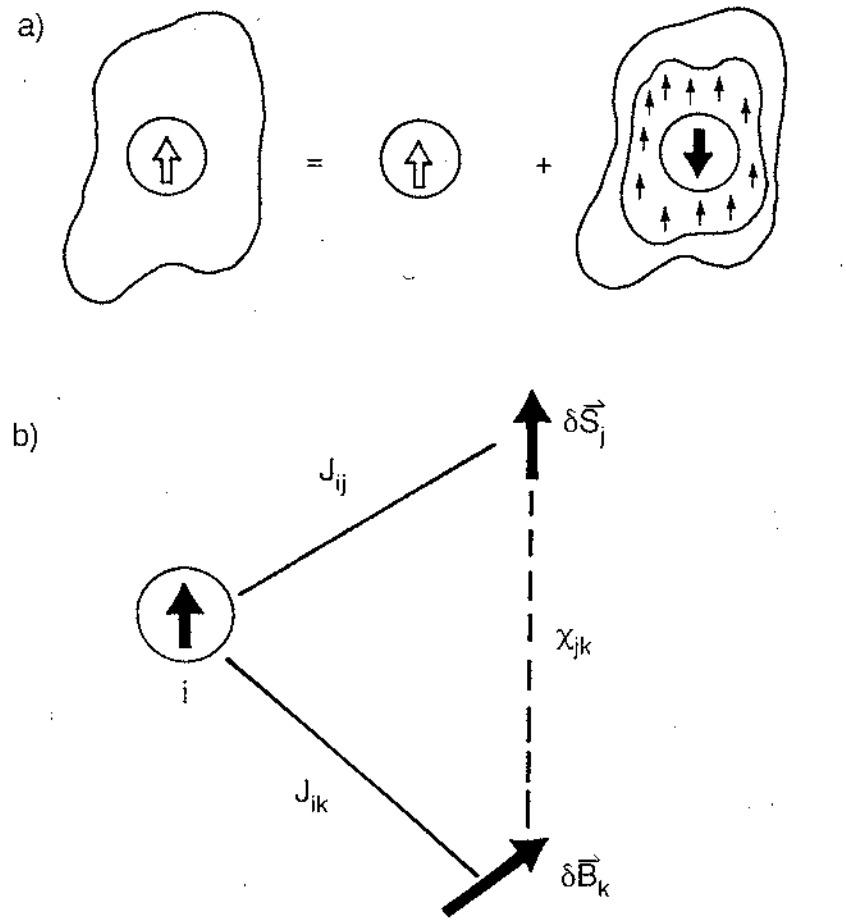

Fig. 7. A pictorial summary of the Onsager reaction field in strongly fluctuating antiferromagnets. (a) The orienting field on a spin (i) is the sum of the the uniform Weiss field (ii) and the cavity reaction field (iii). (b) The three sites with their associated magnetizations and fields used to compute the reaction field within linear response theory.

In the approach to an antiferromagnetic transition the moments fluctuate more and more slowly, and the Onsager field eventually freezes into a constant Weiss exchange field.

From a more microscopic standpoint, the Onsager reaction field may be viewed as a chemical potential for spin. If we regard the Onsager reaction field as the

derivative of an effective Hamiltonian

$$\vec{B}_j = -2\mu\vec{S}_j = \frac{\partial H_{\rm I}}{\partial \vec{S}_j}, \tag{2.58}$$

then it may included into the Hamiltonian as a term of the form

$$H_{\rm I} = -\sum_i \mu_i(\tau)\{\vec{S}_i\cdot\vec{S}_i - S(S+1)\}. \tag{2.59}$$

Equation (2.59) is zero within a Gibbs ensemble of definite spin, but can be identified as the constraint term within the "grand-canonical" Schwinger boson approach. The partition function associated with eq. (2.59) is

$$\mathcal{Z} = \int \mathrm{d}[\mu_i]\ \mathrm{Tr}\left\{\exp -\beta\left(H + \sum_i \mu_i[\vec{S}_i\cdot\vec{S}_i - S(S+1)]\right)\right\}. \tag{2.60}$$

Setting $n_i = 2\vec{S}_i$, we find that eq. (2.60) can be rewritten as

$$\begin{aligned}\mathcal{Z} &= \int \mathrm{d}[\mu_i]\ \mathrm{Tr}\left\{\exp -\beta\left(H + \sum_i \mu_i\{S+\tfrac{1}{2}\}[n_i - 2S]\right)\right\}\\ &= \int \mathrm{d}[\lambda_i]\ \mathrm{Tr}\left\{\exp -\beta\left(H + \sum_i \lambda_i[n_i - 2S]\right)\right\},\end{aligned} \tag{2.61}$$

where $\lambda_i = \mu_i(S+\frac{1}{2})$, so that the constraint term can be rewritten in terms of the reaction field. The corresponding mean-field equation

$$\langle n_b\rangle_\lambda = 2S \tag{2.62}$$

is the microscopic counterpart of the cavity field feed-back equation (2.57) first considered by Brout and Thomas [29]. Conceptually, the Schwinger boson approach provides a microscopic realization of the work of Brout and Thomas [29]; it models a spin system of definite S by a grand-canonical ensemble of spin bosons moving in the background of a fluctuating Onsager reaction field, as we shall see shortly.

3. The antiferromagnet as a quantum fluid: applications

In the last section we developed the conceptual framework for a quantum fluids approach to magnetism, and now it is time to put these ideas to practical use. We

begin with the simple example of the nearest-neighbor bipartite Heisenberg antiferromagnet. The Onsager reaction field was first applied to Heisenberg spin systems by Takahashi [42,43], who used it to extend spin-wave theory to finite temperatures; later Hirsch and Tang [44] developed a similar technique for finite-size lattices. Here we use the rotationally invariant Schwinger boson representation of Arovas and Auerbach [35], working within the operator mean-field approach first studied by Sarker, Jayaprakash, Krishnamurthy and Ma [45]. The spin stiffness and the spin correlation length are calculated as a spin-current response, with good agreement with known scaling results. Next, we extend the two-fluids picture to frustrated magnetism, indicating possible pitfalls in doing so. Since the technical aspects of these developments have been well-documented elsewhere [35–38,46], we will limit our remarks to those tools strictly necessary for the forthcoming discussion of enhanced fluctuations in frustrated magnetism; in particular, in the next section we will use the pairing equations developed here to calculate the Ising transition temperature associated with a fluctuation-stabilized order parameter that is robust to the absence of long-range Néel order.

Before plunging into the microscopics of the two-component fluid treatment of magnetism, let us briefly discuss how such a method encompasses the more conventional semiclassical approach. From this perspective, spin-wave theory is a mean-field theory of the "down" spin boson; it models a spin system of definite S by a Gibbs ensemble of spin bosons where the "up" spin boson has condensed. In short, spin-wave theory represents the partition function

$$\mathcal{Z} = \mathrm{Tr}\left\{P_{n=2s}\, \mathrm{e}^{-\beta H}\right\} \tag{3.1}$$

by

$$\mathcal{Z} = \mathrm{Tr}_{\text{"}b_\downarrow\text{"}}\, \mathrm{e}^{-\beta H_{\text{spin wave}}}, \tag{3.2}$$

where the constraint

$$b_\uparrow^\dagger b_\uparrow = 2S - b_\downarrow^\dagger b_\downarrow \tag{3.3}$$

is *solved at the outset*. By contrast, the Schwinger boson method represents a system of fixed spin by a grand-canonical ensemble of bosons where $\langle S_z\rangle = 0$; this treatment is a mean-field theory of the constraint, where fluctuations in the associated Onsager reaction field become interactions. Since both "up" and "down" spin bosons exist in uncondensed form, it has the possibility of treating states that break the SU(2) spin rotational symmetry without the development of single-particle condensation associated with moment-formation.

3.1. A rotationally invariant treatment of the bipartite magnet

We now turn to the bipartite Heisenberg antiferromagnet with the Hamiltonian

$$H = J\sum_{(i,j)} \vec{S}_i \cdot \vec{S}_j\,, \tag{3.4}$$

where J is a nearest-neighbor exchange coupling. Conventionally, this model has been treated perturbatively around the "large-S" classically ordered Néel state with bosonic spin deviation operators, a method that explicitly breaks the spin rotational symmetry [5,6]. This spin-wave treatment can be recovered from a Schwinger boson approach by eliminating one component of the Bose field using the constraint

$$b_\uparrow^\dagger b_\uparrow = 2S - b_\downarrow^\dagger b_\downarrow. \tag{3.5}$$

Here the "up" spin direction is chosen to coincide with the direction of the magnetization at each site. Since a gauge can always be chosen so that the up spin field is real, the constraint then permits the replacement

$$\left.\begin{matrix} b_\uparrow^\dagger \\ b_\uparrow \end{matrix}\right\} \longrightarrow \sqrt{2S - b_\downarrow^\dagger b_\downarrow}\,, \tag{3.6}$$

so that the physical spin operators become

$$\begin{aligned} S_z &= \tfrac{1}{2}(b_\uparrow^\dagger b_\uparrow - b_\downarrow^\dagger b_\downarrow) \longrightarrow S - b_\downarrow^\dagger b_\downarrow, \\ S_+ &= b_\uparrow^\dagger b_\downarrow \longrightarrow \left(\sqrt{2S - b_\downarrow^\dagger b_\downarrow}\right) b_\downarrow, \\ S_- &= b_\downarrow^\dagger b_\uparrow \longrightarrow b_\downarrow^\dagger \left(\sqrt{2S - b_\downarrow^\dagger b_\downarrow}\right). \end{aligned} \tag{3.7}$$

This is the well-known Holstein–Primakoff representation; from this exercise we see that by imposing the incompressibility constraint at the outset, conventional spin-wave theory immediately breaks the SU(2) spin rotational symmetry of the problem.

As we discussed in the preceding section, a rotationally invariant treatment of quantum antiferromagnetism demands that the local environment of a single spin is described by an Onsager reaction field; formally this reaction field appears as a chemical potential that constrains the density of the spins,

$$H(\tau) = H_0 + \sum \lambda_j(\tau)\,[n_j - 2S]. \tag{3.8}$$

The associated partition function is

$$Z = \int \mathcal{D}[\lambda_j]\ \mathrm{Tr}\ \mathrm{e}^{-\int_0^\beta H(\tau)\,\mathrm{d}\tau}, \tag{3.9}$$

where the integral over the λ fields imposes the constraint $n_j = 2S$; it is taken in the complex plane along a line parallel to the imaginary axis. From a practical standpoint, the incompressibility condition $n_j = 2S$ is imposed on average (i.e. $\langle n_j \rangle = 2S$); from a path-integral approach the Onsager field is the value of $\lambda_j(\tau)$ at the saddle point of the constraint integral. Such an approximation is clearly unreliable at small S, where fluctuations in the constraint field may become increasingly important.

Let us now do a simple Schwinger boson mean-field analysis for the bipartite antiferromagnet, a method first developed by Arovas and Auerbach [35]. We begin by decoupling the original Hamiltonian in terms of singlet pairs

$$\tilde{B}^\dagger_{ij} = b^\dagger_{i\uparrow} b^\dagger_{j\downarrow} - b^\dagger_{i\downarrow} b^\dagger_{j\uparrow} \qquad (i \in A, j \in B), \tag{3.10}$$

where the spins i and j are on the A and B sublattices, respectively. We transform to a staggered spin reference frame

$$\begin{aligned} &b_{j\sigma} \to b_{j\sigma}\ \mathrm{e}^{\mathrm{i}\sigma \vec{Q}\cdot\vec{R}_j}, \\ &\tilde{B}^\dagger_{ij} \to B_{ij} = b^\dagger_{i\uparrow} b^\dagger_{j\downarrow} + b^\dagger_{i\downarrow} b^\dagger_{j\uparrow} \qquad (i \in A, j \in B) \end{aligned} \tag{3.11}$$

by performing a π rotation on all spins on the B sublattice; in this "Marshall" frame, the singlet pairing field is symmetric. The Hamiltonian (3.4) can then be written

$$H = J \sum_{(i,j)} \{(2S)^2 - B^\dagger_{ij} B_{ij}\} + \lambda \sum_j (n_j - 2S). \tag{3.12}$$

Developing a mean-field theory in B_{ij}, we do a Hubbard–Stratonovich factorization of the interaction term of eq. (3.12), using a field

$$\Delta_{ij} \sim -\frac{J}{2}\langle B_{ij} \rangle \tag{3.13}$$

to represent the fluctuations in spin pairing at each bond. Formally, we introduce the Gaussian identity

$$1 = \int \mathcal{D}[\bar{\Delta}_{ij}, \Delta_{ij}] \exp\left[-\int_0^\beta \mathrm{d}\tau\, \frac{2}{J}\left(\bar{\Delta}_{ij} - \frac{J}{2} B^\dagger_{ij}\right)\left(\Delta_{ij} - \frac{J}{2} B_{ij}\right)\right] \tag{3.14}$$

into the path integral representation of the partition function. After expansion, the factorized Hamiltonian takes the form

$$H = \sum_{(i,j)} \left\{ \left[\bar{\Delta}_{ij} B_{ij} + \mathrm{h.c.}\right] + \frac{2\bar{\Delta}_{ij}\Delta_{ij}}{J} \right\} + \sum_j \lambda_j (n_j - 2S). \tag{3.15}$$

Mean-field theory then replaces $\lambda_j \to \lambda$ and $\Delta_{ij} \to \Delta$, where the average values are determined by the stationarity of the mean-field free energy with respect to small variations in both variables.

The corresponding mean-field Hamiltonian is then translationally invariant, and in momentum space can be written

$$\begin{aligned} H_{\mathrm{MFT}} = {} & \lambda \sum_{\vec{q}} (b^\dagger_{\vec{q}\uparrow} b_{\vec{q}\uparrow} + b_{-\vec{q}\downarrow} b^\dagger_{-\vec{q}\downarrow}) \\ & - \sum_{\vec{q}} \Delta_{\vec{q}} (b^\dagger_{\vec{q}\uparrow} b^\dagger_{-\vec{q}\uparrow} + b_{-\vec{q}\downarrow} b_{\vec{q}\downarrow}) - N_s(2S+1), \end{aligned} \tag{3.16}$$

where N_s is the number of sites, and $\Delta_{\vec{q}} = 2\Delta \sum_{l=1,d} \cos q_l$. To diagonalize H_{MFT}, we make the Bogoliubov transformation

$$a^\dagger_{\vec{q}\sigma} = u_{\vec{q}} b^\dagger_{\vec{q}\sigma} - v_{\vec{q}} b_{-\vec{q}-\sigma}, \tag{3.17}$$

which preserves the bosonic commutation properties of the creation operators, and yields the diagonalized Hamiltonian

$$H_{\mathrm{MFT}} = \sum_{\vec{q}} \omega_{\vec{q}\sigma} \left[a^\dagger_{\vec{q}} a_{\vec{q}\sigma} + \frac{1}{2} \right] - N_s \lambda (2S+1) + \frac{Z N_s \Delta^2}{J}, \tag{3.18}$$

with $Z = 2d$ the number of nearest-neighbors per site. Substituting for the $a^\dagger_{\vec{q}\sigma}$, we find that

$$\begin{aligned} \omega_{\vec{q}} &= \sqrt{\lambda^2 - \Delta^2_{\vec{q}}}, \\ u^2_{\vec{q}} &= \frac{1}{2} + \frac{\lambda}{2\omega_{\vec{q}}}, \\ v^2_{\vec{q}} &= \frac{1}{2} - \frac{\lambda}{2\omega_{\vec{q}}}. \end{aligned} \tag{3.19}$$

This harmonic-oscillator form of the Hamiltonian yields the free energy

$$F = 2T \sum_{\vec{q}} \ln[2\sinh(\tfrac{1}{2}\beta\omega_{\vec{q}})] + N_s \left[\frac{Z\Delta}{J} - \lambda(2S+1) \right]. \tag{3.20}$$

By imposing the self-consistency conditions

$$\frac{\partial F}{\partial \Delta} \propto \left\langle \Delta_{ij} + \frac{2\Delta}{J} \right\rangle = 0,$$
$$\frac{\partial F}{\partial \lambda} = \langle n_j - 2S \rangle = 0 \tag{3.21}$$

we arrive at the mean-field equations

$$S = \int_{\vec{q}} \left\{ \frac{\lambda}{\omega_{\vec{q}}} \left[n(\omega_{\vec{q}}) + \frac{1}{2} \right] - \frac{1}{2} \right\},$$
$$\Delta = \frac{J}{Z} \int_{\vec{q}} \frac{1}{\omega_{\vec{q}}} \left[n(\omega_{\vec{q}}) + \frac{1}{2} \right]. \tag{3.22}$$

The first equation in eqs. (3.22) comes from the constraint. For $d > 2$ the Schwinger boson field condenses at the Néel temperature. When $\lambda^2 = (Z\Delta)^2$, the spectrum becomes gapless at $\vec{q} = 0, \vec{Q}$. For the sake of example, let Bose condensation occur at $\vec{q} = 0$, with $\langle b_{j\uparrow} \rangle = \langle b_{j\downarrow} \rangle = \sqrt{M/2}$ corresponding to a staggered magnetization

$$\vec{S}(\vec{R}) = (\cos \vec{Q} \cdot \vec{R}, 0, 0). \tag{3.23}$$

The occupancy and pairing fields now acquire an additional delta-function component related to the Bose condensation

$$\langle n_{\vec{q}\sigma} \rangle = \frac{\lambda}{\omega_{\vec{q}}} \left[n(\omega_{\vec{q}}) + \frac{1}{2} \right] - \frac{1}{2} + \delta_{\vec{q}0} M,$$
$$\langle B^\dagger_{ij} \rangle = \frac{2\Delta}{\omega_{\vec{q}}} \left[n(\omega_{\vec{q}}) + \frac{1}{2} \right] + 2M\delta_{\vec{q}0}, \tag{3.24}$$

and the mean-field equations (3.22) must be rewritten as

$$S = M + \frac{1}{2} \int_{\vec{q}} \left\{ \frac{\lambda}{\omega_{\vec{q}}} \left[n(\omega_{\vec{q}}) + \frac{1}{2} \right] - \frac{1}{2} \right\},$$
$$\Delta = \frac{J}{Z} \left\{ \int_{\vec{q}} \frac{\Delta_{\vec{q}}}{\omega_{\vec{q}}} \left[n(\omega_{\vec{q}}) + \frac{1}{2} \right] + M \right\}. \tag{3.25}$$

In the limit $S \to \infty$ the condensate fraction completely dominates these integral equations,

$$\left.\begin{aligned} \lambda &= ZJS \\ \Delta &= JS \end{aligned}\right\} \; S \to \infty,$$

and the low-energy spin excitations around $\vec{q} = 0$ ($\vec{Q}$) have the form $\omega_{\vec{q}} = cq$ ($c|\vec{q} - \vec{Q}|$) with a spin-wave velocity

$$c = \sqrt{2Z}\, JS. \tag{3.26}$$

In two dimensions the situation is particularly interesting, since here long-wavelength thermal fluctuations have enough phase space to prevent conventional magnetic ordering at any temperature. At low temperatures, the excitation spectrum has a small finite gap

$$\Delta_g(T)^2 = \lambda(T)^2 - (Z\Delta(T))^2, \tag{3.27}$$

and $\omega_q = \sqrt{\Delta_g^2 + c^2 q^2}$ in the vicinity of $\vec{q} = 0$, which leads to a finite spin correlation length $\xi = c/\Delta_g(T)$. At low temperatures the constraint equation is dominated by the thermal component of the integral in the vicinity of $\vec{q} \sim \vec{Q}$; with the approximation $n(\omega) \sim T/\omega$, to logarithmic accuracy it becomes

$$S = \frac{2\lambda}{c^2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{T}{q^2 + \xi^{-2}}. \tag{3.28}$$

Setting $\rho_s = 2JS^2$, we find that

$$\frac{\rho_s}{T} = \frac{1}{4\pi} \ln \left(\frac{\xi}{a} \right) \tag{3.29}$$

or, to exponential accuracy,

$$\xi \approx a \exp \left(\frac{4\pi\rho_s}{T} \right), \tag{3.30}$$

a result consistent with that obtained from a scaling analysis of the equivalent non-linear σ-model [35,17].

Let us proceed a bit further, and study the spin stiffness within this framework. Let us twist the magnet by applying twisted boundary conditions about the z-axis. In order to absorb these new boundary conditions into the Bose creation operators, we must transform $b^\dagger_{\vec{q}\sigma} \to b^\dagger_{\vec{q}+\vec{A}\sigma\sigma}$, where $\vec{A} = \vec{\nabla}\theta$ is the twist vector potential about the z-direction. The "twisted" Hamiltonian is then

$$H_{\mathrm{MFT}}[A] = \lambda \sum_{\vec{q}} (b^\dagger_{\vec{q}\uparrow} b_{\vec{q}\uparrow} + b_{-\vec{q}\downarrow} b^\dagger_{-\vec{q}\downarrow})$$
$$- \sum_{\vec{q}} \Delta_{\vec{q}-\vec{A}} (b^\dagger_{\vec{q}\uparrow} b^\dagger_{-\vec{q}\uparrow} + b_{-\vec{q}\downarrow} b_{\vec{q}\downarrow}) - N_s(2S+1) \tag{3.31}$$

and the uniform z-axis spin current is

$$\begin{aligned}
\vec{J} &= -\nabla_A H[\vec{A}] = \vec{J}^{(\mathrm{p})} + \vec{J}^{(\mathrm{a})}, \\
J_l^{(\mathrm{p})} &= \nabla_{q_l} \Delta_{\vec{q}} (b^\dagger_{\vec{q}\uparrow} b^\dagger_{-\vec{q}\uparrow} + b_{-\vec{q}\downarrow} b_{\vec{q}\downarrow}), \\
J_l^{(\mathrm{a})} &= -\nabla^2_{ll'} \Delta_{\vec{q}} A_{l'} (b^\dagger_{\vec{q}\uparrow} b^\dagger_{-\vec{q}\uparrow} + b_{-\vec{q}\downarrow} b_{\vec{q}\downarrow}).
\end{aligned} \tag{3.32}$$

In analogy with the superfluid case, the two components of the current $\vec{J}^{(\mathrm{p})}$ and $\vec{J}^{(\mathrm{a})}$ are identified as the "paramagnetic" and diamagnetic response, respectively.

The spin stiffness is the linear spin-current response of the spin fluid to the application of a twist. If we write $F[A] = -T \ln \mathrm{Tr}[\exp(-\beta H_{\mathrm{MFT}})]$, and differentiate with respect to $\vec{A}$, then we find that

$$\nabla_{\alpha\beta} F[A] = Q^{(\mathrm{p})}_{\alpha\beta} + Q^{(\mathrm{d})}_{\alpha\beta}, \tag{3.33}$$

where

$$Q^{(\mathrm{p})}_{\alpha\beta} = \langle J_\alpha(\omega) J_\beta(-\omega) \rangle_{\omega=0} \tag{3.34}$$

is the "paramagnetic" response associated with the spin currents carried by spin fluctuations and

$$Q^{(\mathrm{d})}_{\alpha\beta} = \sum_{\vec{q}} \nabla^2_{\alpha\beta} \Delta_{\vec{q}} \langle (b^\dagger_{\vec{q}\uparrow} b^\dagger_{-\vec{q}\uparrow} + b_{-\vec{q}\downarrow} b_{\vec{q}\downarrow}) \rangle$$

is the instantaneous "diamagnetic" response of the entire spin fluid to the application of a twist. As in other quantum fluids, the paramagnetic and diamagnetic spin currents associated with the normal fluid will cancel shortly after the application of a change of boundary conditions, leaving a residual stiffness associated with the condensate. In the large-S limit this residual stiffness is

$$Q^{(\mathrm{d})}_{\alpha\beta} = \sum_{\vec{q}} \nabla^2_{\alpha} \Delta_{\vec{q}} [\langle b^\dagger_{\vec{q}\uparrow} \rangle \langle b^\dagger_{-\vec{q}\uparrow} \rangle + \mathrm{h.c.}] = 2JS^2. \tag{3.35}$$

In a finite-size system in the absence of true Bose condensation the spectral weight associated with the spin condensate is spread out in a narrow region of momentum space of radial extent $\Delta q \lesssim (1/L)$. The associated stiffness is approximately

$$Q^{(\mathrm{d})}_{\alpha\beta} \approx \int_{|\vec{q}|,\, |\vec{q}-\vec{Q}|<1/L} \frac{\mathrm{d}^2 q}{(2\pi)^2} \nabla^2_{\alpha\beta} \Delta_{\vec{q}} \frac{\Delta_{\vec{q}}}{\omega_{\vec{q}}} \left[\frac{1}{2} + n(\omega_{\vec{q}}) \right], \tag{3.36}$$

which yields

$$\begin{aligned}
Q^{(\mathrm{d})}_{\alpha\beta} &= \rho_s \delta_{\alpha\beta} \approx \delta_{\alpha\beta} \left(\nabla^2_{\alpha\beta} \Delta_{\vec{q}} \right)_{\vec{q}=0} \int_{|\vec{q}|<1/L} \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{T}{(cq)^2 + \Delta_{\mathrm{g}}^2} \\
&= \frac{T}{4\pi} \ln \left[1 + \frac{\xi^2}{L^2} \right].
\end{aligned} \tag{3.37}$$

Thus, as the system size L exceeds the spin correlation length, fluctuations in the spin field drive the spin stiffness to zero. This phenomenon can be more quantitatively studied in the context of non-linear σ-models for antiferromagnets. The quantity

$$g = \frac{T}{\rho_s(L)} \approx \frac{4\pi}{\ln[1 + \xi^2/L^2]} \tag{3.38}$$

can be viewed as a coupling constant between spin fluctuations; when the spin stiffness is large, spin-wave fluctuations are essentially non-interacting, whereas when it is small, such is no longer the case. In this respect, 2D spin fluctuations are weakly interacting on length scales (L) less than ξ, but interact strongly for $L > \xi$. Within the quantum fluids approach, a more rigorous way of deriving this result is to examine the momentum dependence of the spin-current response [38].

So, we see that the stiffness of a quantum antiferromagnet can be interpreted as a spin-current response with the two-fluids perspective. As emphasized in the last section, the presence of a spin stiffness does not imply spin supercurrents, a condition that relies on the topological stability of spin vortices. Nonetheless, we can learn about the stiffness of a quantum antiferromagnet by treating the spin-current response to an external twist in a fashion analogous to that used in conventional quantum fluids. Equations (3.32) indicate that the stiffness of a quantum antiferromagnet is essentially a *local* response function; this link between the spin stiffness and local spin-current fluctuations is a key consequence of the quantum fluids approach to magnetism. From a practical standpoint, this relationship is very useful in numerical calculations of the stiffness in classical Heisenberg magnets. More specifically, it is more accurate to monitor local current fluctuations, rather than to compute the free energy as a function of external boundary conditions, which requires several Monte Carlo runs and modified code. As an additional check on our quantum fluids approach, Ritchey [13] has used the local spin-current fluctuations as a means to probe the stiffness of the classical two-dimensional Heisenberg antiferromagnet on a square lattice; his measured spin stiffness, γ/T, is plotted as a function of T and L in fig. 8. From this data he estimates the linear dependence of γ/T on $\ln L$ to be

$$\frac{\mathrm{d}(\gamma/T)}{\mathrm{d} \ln L} = \frac{1}{\ln_2 L} \frac{\mathrm{d}(\gamma/T)}{\mathrm{d} \ln_2 L} \sim -0.12 \tag{3.39}$$

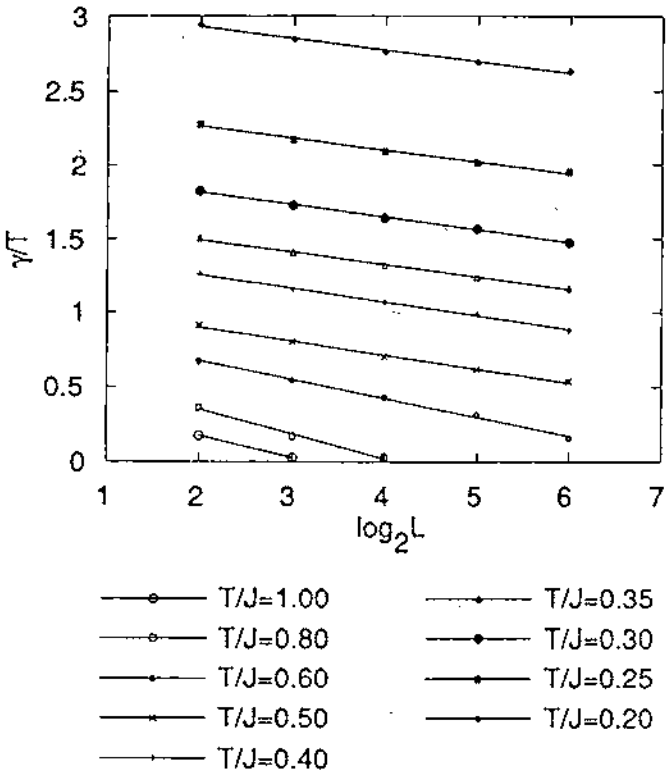

Fig. 8. Spin stiffness of the square-lattice antiferromagnet calculated by Ritchey [13] in Monte Carlo simulations, where the solid lines are least-squares fits to the data.

from a least-squares fit, which is in good quantitative agreement with the Polyakov scaling equation [47] for the O(3) non-linear σ-model

$$\frac{\mathrm{d}(\gamma/T)}{\mathrm{d}\ln L} = -\frac{1}{2\pi} \sim -0.16, \tag{3.40}$$

thus providing further confidence in this type of analysis. We note that eq. (3.40) assumes that the cubic corrections to the scaling are negligible, and thus we expect numerical agreement to be exact only in the limit of $\gamma/T \to \infty$.

3.2. *Frustrated magnetism from a two-fluids perspective*

3.2.1. *Difficulties in non-collinear spin systems*

Anderson has always emphasized the *singlet* nature of spin pairing in his fluids approach to magnetism; from this standpoint the spin correlations will always be

isotropic in character. For bipartite lattices, Arovas and Auerbach (AA) [35] have showed that a mean-field treatment of these singlet bonds is formally exact within an SU(N) generalization; in this large-N limit pairing fluctuations are quenched by the number of flavors. AA noted that if one relabels the Bose fields on the odd sublattices and then permits an infinite number of spin variables, the bond operators behave in a semiclassical fashion. Following this prescription, the Heisenberg Hamiltonian can be decoupled entirely using singlet pairs; for the ferromagnet

$$\vec{S}_i \cdot \vec{S}_j = :\tfrac{1}{2} D^\dagger_{ij} D_{ij}: - S^2 \tag{3.41}$$

and for the antiferromagnet

$$\vec{S}_i \cdot \vec{S}_j = -\tfrac{1}{2} B^\dagger_{ij} B_{ij} + S^2, \tag{3.42}$$

where

$$\begin{aligned} D^\dagger_{ij} &= b^\dagger_{i\sigma} b_{j\sigma}\,, \\ B^\dagger_{ij} &= (b^\dagger_{i\sigma} b^\dagger_{j-\sigma})\sigma \end{aligned} \tag{3.43}$$

are particle–hole and Cooper singlet pairing fields, respectively.

Clearly the Arovas–Auerbach prescription [35] will not work for non-collinear spin systems, where both ferromagnetic and antiferromagnetic interactions are present. How to extend this quantum fluids perspective to the case of incommensurate or frustrated spin structures? One group, led by Read and Sachdev [37] and Yoshioka and Miyazaki [48], preserves the *singlet* character of the macroscopic wavefunction, describing the microscopic interactions in terms of *isotropic* pairing fields. Extending the Arovas–Auerbach decoupling scheme to non-collinear magnets, Read and Sachdev [49] have introduced an additional flavor index to the Schwinger boson; the new singlet pair operator is now the sum of N replicas

$$B^\dagger_{ij} \to \tilde{B}^\dagger_{ij} = \sum_{\lambda=1}^{N} B^{\dagger(\lambda)}_{ij} = b^\dagger_{i\alpha} J^{\alpha\beta} b^\dagger_{j\beta}\,, \tag{3.44}$$

where

$$J^{\alpha\beta} = J^{(\sigma\lambda)(\sigma'\lambda')} \equiv \sigma \delta_{\sigma,-\sigma'} \delta_{\lambda\lambda'}\,, \tag{3.45}$$

which again behaves semiclassically in the large-N limit; Yoshioka and Miyazaki have treated the triangular lattice problem in a similar fashion [48]. However, in both cases the mean-field spectrum has the general form

$$\omega_{\vec{q}} = \sqrt{\lambda^2 - \Delta^2_{\vec{q}}}, \tag{3.46}$$

which, unlike the AA dispersion for bipartite lattices, does *not* recover the semiclassical result from spin-wave theory

$$\omega_{\vec{q}} = \sqrt{\gamma_{\vec{q}}^2 - \Delta_{\vec{q}}^2}. \tag{3.47}$$

In this approach there are no anisotropic fluctuations in the "normal" fluid by construction; thus this method will not be sensitive to the formation of tensor spin order in the absence of a sublattice magnetization.

By contrast, the approach we will describe here takes its cue from helium-4; there, though in principle one could project out the singlet component of the ground state, the important effects of anisotropy have been incorporated by including microscopic *triplet* pairing [33]. Analogously, the fluctuation anisotropy of non-collinear magnets can be emphasized by permitting *both* singlet and triplet pairing in the normal fluid [38]. From another perspective, the presence of a twist breaks inversion symmetry and thus violates parity; microscopically

$$\vec{S}_i \times \vec{S}_j = \tfrac{1}{4}(B_{ij}^\dagger \vec{B}_{ij} + \text{h.c.}), \tag{3.48}$$

where $B_{ij}^\dagger$ and $\vec{B}_{ij}^\dagger = b_{i\alpha}^\dagger (\vec{\sigma}\sigma^2)_{\alpha\beta} b_{i\beta}^\dagger$ are the singlet (see eq. (3.43)) and the triplet pairing fields between sites i and j. A *unique* decoupling of the Heisenberg Hamiltonian is performed in a "Marshall" reference frame, one with *only* even-parity spin pairing fields; in the large-S limit the calculated dispersion relation agrees with the semiclassical spin-wave theory result [38].

3.2.2. Gauge-fixing and the "Marshall" reference frame

In the previous section, we discussed a gauge-invariant formulation of the Heisenberg model; let us now return to this topic to formally indicate the transformation to the "Marshall" reference frame where the parity-violating twist vanishes. Physically, because any *arbitrary* magnetic configuration can be described by the action of twist(s) on a locally ferromagnetic configuration, all calculations can be performed in a twisted coordinate system where the magnet is treated as an even-parity, triplet-paired Bose fluid. For simplicity, here we will consider an equilibrium magnetic structure that is uniformly twisted (i.e. one twist), and our discussion here follows closely our treatment of this topic elsewhere [38].

In a Lagrangian formalism, the partition function associated with the Heisenberg model is

$$Z = \int_{\lambda_0}^{\lambda_0 + 2i\pi T} \mathrm{d}\lambda_j \int \mathcal{D}[b]\, \mathrm{e}^{-\int \mathcal{L}(\tau)\,\mathrm{d}\tau},$$
$$\mathcal{L} = \mathcal{L}_0 + H, \qquad \mathcal{L}_0 = \sum_j \{b_j^\dagger [\partial_\tau - \lambda_j] b_j + 2S\lambda_j\}, \tag{3.49}$$

where H is the Hamiltonian in terms of Schwinger bosons. The fluctuating Onsager potential λ_j imposes the constraints, generating local Onsager cavity fields. Spin indices on the Bose fields have been suppressed ($b_j^\dagger = (b_{j\uparrow}^\dagger, b_{j\downarrow}^\dagger)$).

Under an independent rotation of the spin basis at each site

$$b_j^\dagger = b_j'^\dagger g_j\,,$$
$$g_j = \mathrm{e}^{(i/2)\vec{\theta}_j \cdot \vec{\sigma}} \tag{3.50}$$

the spin transforms under the adjoint representation of SU(2)

$$\vec{S}_j' = \tfrac{1}{2} b_{j\alpha}'^\dagger \vec{\sigma}_{\alpha\beta} b_{j\beta}' = \mathrm{e}^{-\vec{\theta}_j \times} \vec{S}_j\,. \tag{3.51}$$

The transformed Lagrangian is then $\mathcal{L}^g = \mathcal{L}_0 + H^g$, where

$$H^g = \tfrac{1}{2} \sum J_{ij} \vec{S}_i\, \mathrm{e}^{-\vec{A}_{ij} \times} \vec{S}_j - \sum_j \vec{B}_j \cdot \vec{S}_j \tag{3.52}$$

and primes have been dropped for clarity. The exponential $\mathrm{e}^{-\vec{A}_{ij}\times}$ is shorthand for the O(3) rotation matrix

$$(\mathrm{e}^{-\vec{A}_{ij}\times})^{pq} = (\mathrm{e}^{-\vec{\theta}_i \times}\ \mathrm{e}^{\vec{\theta}_j \times})^{pq}. \tag{3.53}$$

If we gauge transform the Bose fields $b \to b^g$ inside the path integral (3.49), then the Lagrangian becomes $\mathcal{L} \to \mathcal{L}^g$. The integration measure is gauge-invariant, $\mathcal{D}[b] = \mathcal{D}[b^g]$, as is the partition function

$$Z = Z^g = \int_{\lambda_0}^{\lambda_0 + 2i\pi T} \mathrm{d}\lambda_j \int \mathcal{D}[b]\, \mathrm{e}^{-\int \mathcal{L}^g(\tau)\,\mathrm{d}\tau}. \tag{3.54}$$

We can average over gauges, using a normalized weighting function $F(g)$: $\int \mathcal{D}[g]\, F(g) = 1$. We then write $Z = \int \mathrm{d}g\, F(g) Z^g$, or

$$Z = \int_{\lambda_0}^{\lambda_0 + 2i\pi T} \mathrm{d}\lambda_j \int \mathcal{D}[g, b]\, F(g)\, \mathrm{e}^{-\int \mathcal{L}^g(\tau)\,\mathrm{d}\tau}. \tag{3.55}$$

We now select the weighting function that simplifies the decoupling procedure, one that allows us to work in a twisted reference frame with only even-parity spin pairing. Formally, this is equivalent to integrating over the spin gauge fields and imposing a gauge-fixing condition on the spin configurations

$$\langle \vec{S}_i^g \times \vec{S}_j^g \rangle = 0. \tag{3.56}$$

The transformed Lagrangian now becomes

$$\mathcal{L}^{\mathrm{g}} = \mathcal{L}_0^{\mathrm{g}} + H_0^{\mathrm{g}} + \sum (J_{ij} \sin\theta_{ij}\, \hat{k}_{ij} - \vec{\lambda}_{ij}) \cdot (\vec{S}_i \times \vec{S}_j), \tag{3.57}$$

where an integral over the Lagrange multipliers $\vec{\lambda}_{ij}$ fixes the average gauge; in this "Marshall" frame the twist vanishes and there exist only even-parity spin pairing fields.

3.2.3. The mean-field decoupling procedure

The gauge-fixing condition described above ensures that at the saddle point the Hamiltonian will contain no parity-mixing terms. We now make a pairing Ansatz for the triplet Cooper and the singlet particle–hole pairing fields:

$$\begin{aligned} \langle \vec{B}^{\dagger}_{\vec{q}} \rangle &= \langle b^{\dagger}_{\vec{q}\alpha} (\vec{\sigma}\sigma^2)_{\alpha\beta}\, b^{\dagger}_{-\vec{q}\beta} \rangle = 2\eta_{\vec{q}}\, \hat{k}, \\ \langle D^{\dagger}_{\vec{q}} \rangle &= \langle b^{\dagger}_{\vec{q}\uparrow} b_{\vec{q}\uparrow} + b_{\vec{q}\downarrow} b^{\dagger}_{\vec{q}\downarrow} \rangle = 2\alpha_{\vec{q}}, \end{aligned} \tag{3.58}$$

where $\hat{k}$ is the twist axis and $\eta_{\vec{q}}$ and $\alpha_{\vec{q}}$ are even functions of $\vec{q}$. In the large-S classical limit ($S \to \infty$), these pairing correlations become $\alpha_{\vec{q}} = \eta_{\vec{q}} = S\delta_{\vec{q}} + \mathrm{O}(1)$.

At the saddle point, the twisted Hamiltonian is then

$$H = \tfrac{1}{2} \sum_{ij} J_{ij} \{ \gamma^{(+)} \vec{S}_i \cdot \vec{S}_j + \gamma^{(-)} [\vec{S}_i \cdot \vec{S}_j - 2(\vec{S}_i \cdot \hat{k})(\vec{S}_j \cdot \hat{k})] \}, \tag{3.59}$$

where $\gamma^{(\pm)} = \frac{1}{2}[1 \pm \cos(\vec{Q} \cdot \vec{R}_{ij})]$. The terms in the Hamiltonian can now be decoupled into even-parity pairs

$$\begin{aligned} \vec{S}_i \cdot \vec{S}_j &= \tfrac{1}{2} {:}D^{\dagger}_{ij} D_{ij}{:} - S^2, \\ \vec{S}_i \cdot \vec{S}_j - 2(\vec{S}_i \cdot \hat{k})(\vec{S}_j \cdot \hat{k}) &= \tfrac{1}{2} B^{(0)\,\dagger}_{ij} B^{(0)}_{ij} - S^2, \end{aligned} \tag{3.60}$$

where $B^{(0)\,\dagger}_{ij} = \hat{k} \cdot \vec{B}^{\dagger}_{ij}$. Restricting our attention to zero-momentum pairing we can write the coupling as a BCS Hamiltonian

$$H_{\mathrm{BCS}} = \tfrac{1}{4} \sum_{\vec{q}\vec{q}'} [\mathcal{J}^{+}_{\vec{q}\vec{q}'} D^{\dagger}_{\vec{q}} D_{\vec{q}'} - \mathcal{J}^{-}_{\vec{q}\vec{q}'} B^{(0)\,\dagger}_{\vec{q}} B^{(0)}_{\vec{q}'}] - \tfrac{1}{2} N J(\vec{Q}) S^2. \tag{3.61}$$

Here N is the number of sites and the pairing potentials are

$$\mathcal{J}^{\pm}_{\vec{q}\vec{q}'} = \tfrac{1}{2} \{ J(\vec{q} + \vec{q}') \pm \tfrac{1}{2} [J(\vec{q} + \vec{q}' + \vec{Q}) + J(\vec{q} + \vec{q}' - \vec{Q})] \}_{\mathrm{S}}. \tag{3.62}$$

The subscript S denotes symmetrization with respect to q and q'. Here, the first and second terms are the pure ferromagnetic and antiferromagnetic pairing potentials, respectively.

In order to study the mean-field theory associated with eq. (3.61) we make the standard Ansatz

$$\begin{aligned} D^{\dagger}_{ij} &= \sum \langle D^{\dagger}_{q} \rangle\, \mathrm{e}^{\mathrm{i}q(R_i - R_j)} + \delta D^{\dagger}_{ij}, \\ B^{\dagger}_{ij} &= \sum \langle B^{\dagger}_{q} \rangle\, \mathrm{e}^{\mathrm{i}q(R_i - R_j)} + \delta B^{\dagger}_{ij}. \end{aligned} \tag{3.63}$$

Defining

$$\begin{aligned} h_q &= \tfrac{1}{2} \sum \mathcal{J}^{+}_{qq} \langle D_q \rangle + \lambda, \\ \Delta_q &= \tfrac{1}{2} \sum \mathcal{J}^{-}_{qq} \langle B_q \rangle + \lambda, \end{aligned} \tag{3.64}$$

the resulting mean-field Hamiltonian is

$$\begin{aligned} H_{\mathrm{MF}} = \sum_{\vec{q}} \{ &(h_{\vec{q}} - \lambda)[b^{\dagger}_{\vec{q}\uparrow} b_{\vec{q}\uparrow} + b_{-\vec{q}\downarrow} b^{\dagger}_{-\vec{q}\downarrow}] - [\Delta_{\vec{q}} b^{\dagger}_{\vec{q}\uparrow} b^{\dagger}_{-\vec{q}\downarrow} + \mathrm{h.c.}] \} \\ &+ E_{\mathrm{c}} + 2N\lambda(S + \tfrac{1}{2}), \end{aligned} \tag{3.65}$$

where we chose $\hat{k} = \hat{z}$, and

$$E_{\mathrm{c}} = \sum_{\vec{q}\vec{q}'} \{ \Delta_{\vec{q}} [\mathcal{J}^{-}]^{-1}_{\vec{q}\vec{q}'} \Delta_{\vec{q}'} - h_{\vec{q}} [\mathcal{J}^{+}]^{-1}_{\vec{q}\vec{q}'} h_{\vec{q}'} \} - \tfrac{1}{2} N J(\vec{Q}) S^2 \tag{3.66}$$

is the spin condensate energy. The quasiparticle energies of H_{MF} are $\omega_{\vec{q}} = \sqrt{(\tilde{h}_{\vec{q}})^2 - \Delta^2_{\vec{q}}}$, where $\tilde{h}_{\vec{q}} = h_{\vec{q}} - \lambda$, so the total mean-field free energy per unit cell is

$$F = \sum_{\vec{q}} 2T \ln[2 \sinh(\tfrac{1}{2}\beta\omega_{\vec{q}})] + E_{\mathrm{c}} + N\lambda(2S + 1). \tag{3.67}$$

Differentiating with respect to $h_{\vec{q}}$, $\Delta_{\vec{q}}$, and λ yields the mean-field equations

$$\begin{aligned} h_{\vec{q}} &= \int_{\vec{q}'} \mathcal{J}^{+}_{\vec{q}\vec{q}'} \alpha_{\vec{q}'}, \\ \Delta_{\vec{q}} &= \int_{\vec{q}'} \mathcal{J}^{-}_{\vec{q}\vec{q}'} \eta_{\vec{q}'}, \\ S + \tfrac{1}{2} &= \int_{\vec{q}} \alpha_{\vec{q}}, \end{aligned} \tag{3.68}$$

where $\int_{\vec{q}} \equiv \int d^2q/(2\pi)^2$, and

$$(2\alpha_{\vec{q}},\ 2\eta_{\vec{q}}) = (\langle D^{\dagger}_{\vec{q}}\rangle, \langle B^{(0)\,\dagger}_{\vec{q}}\rangle) = [\coth(\tfrac{1}{2}\beta\omega_{\vec{q}})/\omega_{\vec{q}}]\,(\tilde{h}_{\vec{q}}, \Delta_{\vec{q}}). \tag{3.69}$$

For ferromagnets ($\vec{Q} = 0$), the pairing potential $\mathcal{J}^-$ vanishes, and eqs. (3.68) reproduce Takahashi's equations [42]; in the case of collinear antiferromagnets ($\vec{Q} = (\pi, \pi)$) $\mathcal{J}^+ = 0$, and eqs. (3.68) agree with the Arovas–Auerbach result [35].

It is straightforward to recover the semiclassical dispersion relation in the large-S, $T = 0$ limit of these equations; there the bosons condense at $\vec{q} = 0$, and there is a pole in the occupation functions $\alpha_{\vec{q}} \sim \eta_{\vec{q}} \sim S^*\delta_{\vec{q}}$ corresponding to a finite magnetization S^*. As $S \to \infty$, the pole dominates, and $S^*/S \to 1$, so

$$\begin{aligned} h_{\vec{q}} &= S\mathcal{J}^+_{\vec{q}0}, \\ \Delta_{\vec{q}} &= S\mathcal{J}^-_{\vec{q}0}, \end{aligned} \tag{3.70}$$

λ adopts the smallest value consistent with $\omega_0 = 0$, yielding $\lambda = SJ(\vec{Q})$. Thus the dispersion in the $S \to \infty$ limit is

$$\omega^2_{\vec{q}} = S^2[J(\vec{q}) - J(\vec{Q})]\{\tfrac{1}{2}[J(\vec{q}+\vec{Q}) + J(\vec{q}-\vec{Q})] - J(\vec{Q})\}, \tag{3.71}$$

which is identical with the calculated spin-wave spectrum of the twisted magnet [50]. This result then gives us confidence in the technique; we are now ready to proceed beyond the realm of conventionally ordered magnets. In the next section we use eqs. (3.68) to determine the Ising transition temperature associated with a fluctuation-generated order parameter that is robust to the *absence* of long-range Néel order; scaling arguments and numerical results agree with the predicted value for T_{Ising}.

4. Frustration from a distance: the long-wavelength consequences

Magnetic frustration is a short-wavelength phenomenon intimately connected with the geometry of the host lattice. Does it affect the long-distance behavior of a spin system? How does frustration "interact" with strong fluctuations? Conventionally we associate fluctuations, both thermal and quantum, with disorder; for example, the development of a sublattice magnetization is suppressed in a two-dimensional Heisenberg antiferromagnet at finite temperatures. The role of fluctuations is often enhanced in geometrically frustrated spin systems, where all spin bonds cannot be satisfied simultaneously due to the underlying lattice structure. In "weakly" frustrated antiferromagnets the energy associated with each elementary plaquette can

be minimized; by contrast, "strongly" frustrated spin systems have a highly degenerate ground-state manifold. *Short-wavelength* fluctuations now become very important [23–25]: the *anisotropic* fluctuation free energy may be minimized in selected lattice symmetry breaking spin configurations, thus lifting the ground-state degeneracy and possibly stabilizing new forms of spin order.

It is the interplay between strong fluctuations and competing interactions that leads to this "order from disorder", and in this section we shall discuss a specific example of this somewhat counterintuitive phenomenon. We begin with a long-wavelength description of two-dimensional *non-collinear* Heisenberg magnets, following Polyakov's method [47] to determine the scaling equations. This approach implicitly assumes that *only* long-wavelength fluctuations affect the spin stiffness; next we discuss a specific example where, by contrast, fluctuations on the length-scale of the lattice play a crucial role, leading to a fluctuation-stabilized order parameter that can survive the loss of a sublattice magnetization. The relevant Ising transition temperature is calculated using the quantum fluids approach, and is confirmed by both scaling and numerical results.

4.1. The classical long-wavelength action

We begin with a discussion of the long-wavelength spin fluctuations in two-dimensional Heisenberg spin systems. At finite temperatures, temporal quantum fluctuations in these systems are incoherent on time scales $t \sim \hbar/k_B T$ and thus the behavior of (finite temperature) 2D quantum antiferromagnets is governed by a classical action with renormalized parameters that describe the slow spatial modes of the order parameter. In a long-wavelength description of spin systems, we must relate the spin order to an appropriate local response function; the most natural choice is the local spin susceptibility. Broken local spin rotational symmetry results in an *anisotropic* local spin susceptibility, as described by the matrix

$$\underline{\chi}^{ab}(x) = -\frac{\delta^2 F[B]}{\delta B^a(x)\delta B^b(x)}. \tag{4.1}$$

In a uniaxial (collinear) magnet, for example, the spin susceptibility parallel to the magnetization axis $\hat{n}(x)$ vanishes, so that eq. (4.1) has the form

$$\underline{\chi}^{ab}(x) = \chi[\delta^{ab} - \hat{n}^a(x)\hat{n}^b(x)]. \tag{4.2}$$

By contrast, in helical magnets the microscopic magnetization $\vec{S}(x)$ precesses in space; it is given by

$$\vec{S}(x) = M[\cos(\vec{Q}\cdot\vec{x})\,\hat{e}_1 + \sin(\vec{Q}\cdot\vec{x})\,\hat{e}_2], \tag{4.3}$$

where $\vec{Q}$ is the magnetic vector, and $\hat{e}_1$ and $\hat{e}_2$ are two axes orthogonal to the spin precession axis $\hat{e}_3$. Here the long-wavelength response is coarse-grained on length scales greater than the magnetic pitch $\lambda = 2\pi/Q$; the (coarse-grained) susceptibility is *anisotropic*:

$$\underline{\chi} = \chi_\perp[\hat{e}_1\hat{e}_1 + \hat{e}_2\hat{e}_2] + \chi_\parallel \hat{e}_3\hat{e}_3\,, \tag{4.4}$$

with different values parallel ($\chi_\parallel$) and perpendicular ($\chi_\perp$) to the twist axis $\hat{e}_3$. There is a large class of possible tensor spin order parameters [51] that could also lead to this behavior at long distances. Strictly speaking, our discussion is *not* confined to a conventional antiferromagnet with a staggered magnetization; we only require the presence of a local order parameter that breaks spin rotational symmetry. For simplicity we will use "antiferromagnet" in the broadest possible sense, referring to spin systems where the SU(2) Heisenberg symmetry has been broken without or with the development of a local moment. The details of the microscopic order will, however, be important for the classification of allowed defect structures [26]; we shall return to this point in section 6.

As we discussed earlier, the long-wavelength properties of a finite-temperature quantum spin system are determined classically by its slow spatial modes. These modes, in turn, are related to the energy associated with slow variations in the susceptibility orientation. At any spatial point, the local order can be described by

$$\hat{e}_a(x) = \underline{g}(x)\,\hat{e}_a(0) \qquad (a = 1,2,3), \tag{4.5}$$

where $g(x)$ is an O(3) rotation matrix and $\hat{e}_a(0)$ is aligned along the x, y, and z axes, $[\hat{e}_a(0)]_\lambda = \delta_{a\lambda}$, so that the ath row of g represents the vector $\hat{e}_a$,

$$\underline{g}(x) = \begin{bmatrix} \cdots & \hat{e}^1 & \cdots \\ \cdots & \hat{e}^2 & \cdots \\ \cdots & \hat{e}^3 & \cdots \end{bmatrix}. \tag{4.6}$$

It is more convenient to express the rotation matrix $g(x)$ in terms of the generators of rotation. For vectors, these generators are in the adjoint representation, and we write

$$g(x) = \exp\left[\underline{\theta}\right] \qquad (\underline{\theta} = \theta_a \underline{t}_a), \tag{4.7}$$

where

$$\left[\underline{t}_\lambda\right]_{\alpha\beta} = \epsilon_{\alpha\lambda\beta} \tag{4.8}$$

satisfies the SU(2) algebra $[\underline{t}_a, \underline{t}_b] = \epsilon_{abc}\underline{t}_c$. Multiplication by the matrix $\underline{\theta}$ has the effect of vector multiplication: $\underline{\theta}\vec{b} = \vec{\theta} \times \vec{b}$, so $\underline{\theta} \equiv \vec{\theta} \times$, and

$$g(x) = \exp[\vec{\theta} \times\ ]. \tag{4.9}$$

The most general long-wavelength action for the antiferromagnet is

$$S = \frac{F}{T} = \frac{1}{2}\sum_{(a=1,2,3)} \int \mathrm{d}^2x\, K_a \nabla_l \hat{e}_a \cdot \nabla_l \hat{e}_a\,, \tag{4.10}$$

where the quantities $K_a = k_a/T$ are the rigidity of the order parameter about each axis. Using the rotation matrix $g(x) = (\hat{e}_1, \hat{e}_2, \hat{e}_3)$, we can rewrite eq. (4.10) more compactly as

$$S = \tfrac{1}{2}\int \mathrm{d}^2x\ \mathrm{Tr}\left[\underline{K}\nabla_l g^{-1}(x)\nabla_l g(x)\right], \tag{4.11}$$

where $\underline{K}_{ab} = \delta_{ab}K_a$; the form (4.11) of the long-wavelength action is called the non-linear σ-model.

The uniaxial limit of eq. (4.10), where $K_1 = K_2 = 0$ and $K_3 = I$, is appropriate for collinear antiferromagnets; now eq. (4.10) simplifies to a O(3) non-linear σ-model

$$S = \frac{I}{2}\int \mathrm{d}^2x\,(\vec{\nabla}\hat{n})^2 \qquad (\hat{n} \equiv \hat{e}_3). \tag{4.12}$$

Here, the energy is invariant to local rotations about $\hat{e}^3$ due to the uniaxial symmetry present; this leads to a local gauge symmetry that must be treated separately from the other rotation modes. We shall see that this uniaxial fixed point is repulsive, and that a general, non-collinear antiferromagnet scales towards an isotropic limit where

$$S = \frac{K}{2}\int \mathrm{d}^2x\ \mathrm{Tr}[\nabla g^{-1}\nabla g], \tag{4.13}$$

with $K_1 = K_2 = K_3 = K$. As an aside, we note that eq. (4.13) has both a "left"- and a "right"-handed rotation symmetry ($g(x) \to hg(x)$ and $g(x) \to g(x)h$), and is thus often called the SU(2) $\times$ SU(2) chiral model. The matrix g can also be expressed as an SU(2) matrix $g = n_0 + \mathrm{i}\vec{n}\cdot\vec{\sigma}$, where $(n_0, \vec{n})$ is a four-dimensional unit vector ($n_0^2 + \vec{n}^2 = 1$); in this way, one can show that eq. (4.13) is equivalent to the O(4) non-linear σ-model.

Using an analogy with tops, we can develop a very simple physical picture of antiferromagnets at long-wavelengths. The three vectors $\hat{e}_\lambda$ ($\lambda = 1,2,3$) are "body axes" of the magnetic order parameter; they are a function of position *and* time. As in the study of tops, we can "watch" the order parameter as a function of time either in the lab frame or in a reference frame that moves with the body axes. From this standpoint, the non-linear σ-model (4.13) is very convenient, for

it permits a simple transformation between these two frames of reference. For example, a rotation of the order parameter in the lab frame corresponds to "left multiplication" of $g(x)$ by a fixed rotation matrix h,

$$g(x) \longrightarrow h_{\rm L} g(x), \tag{4.14}$$

a transformation which leaves the action invariant. Alternatively, one can rotate the order parameter in the "body" frame by a "right multiplication"

$$g(x) \longrightarrow g(x) h_{\rm R}\,, \tag{4.15}$$

though this is only a symmetry of the antiferromagnet in the special case when the stiffness tensor $\underline{K}$ is isotropic.

To fully exploit the antiferromagnet as a "top in space-time", we must reexpress the action in terms of angular velocities of the order parameter. We define the angular velocity $\nabla_\mu g(x) = g(x)\underline{\omega}_\mu$, or

$$\begin{aligned}
&\underline{\omega}_\mu \equiv \vec{\omega}_\mu \times \; = g^{-1}(x)\nabla_\mu g(x) = -(\nabla_\mu g^{-1}(x))g(x),\\
&\underline{\omega}_\mu = \omega^a_\mu \underline{t}_a\,,\\
&\vec{\omega}_\mu = \omega^a_\mu \hat{e}^a(0),
\end{aligned} \tag{4.16}$$

where the first identity follows from $\nabla_l(g^{-1}g) = \nabla_l(g^{-1})g + g^{-1}\nabla_l(g) = 0$. With this definition $\nabla_\mu \hat{e}_a(x) = g(x)[\vec{\omega}_\mu \times e_a(0)]$, we can identify $\vec{\omega}^\mu$ as the angular velocity of the order parameter, measured in the reference frame of the body axes.

We can now recast the long-wavelength action of the antiferromagnet eq. (4.10) as that of a top. Using eq. (4.16), we write eq. (4.11) as

$$S = \tfrac{1}{2}\int {\rm d}^2x\ {\rm Tr}\left[\underline{K}\,\underline{\omega}_l\underline{\omega}_l\right]. \tag{4.17}$$

Defining $\underline{\omega}_l \equiv \vec{\omega}_l \times$, we have

$$\begin{aligned}
{\rm Tr}\left[\underline{K}\,\underline{\omega}_l\underline{\omega}_l\right] &= -\,{\rm Tr}\left[\vec{\omega}_l \times (\vec{\omega}_l \times \underline{K})\right]\\
&= {\rm Tr}\left[(\vec{\omega}_l \cdot \vec{\omega}_l)\underline{1} - \vec{\omega}_l\vec{\omega}_l]\underline{K}\right]\\
&= \vec{\omega}_l \underline{I} \vec{\omega}_l\,,
\end{aligned} \tag{4.18}$$

where

$$\underline{I} = {\rm Tr}\left[\underline{K}\right]\underline{1} - \underline{K} \tag{4.19}$$

is the moment of inertia tensor; explicitly

$$\underline{I}_{ab} = I_a \delta_{ab}\,, \tag{4.20}$$

where

$$I_a = K_b + K_c \qquad (b \neq c \neq a). \tag{4.21}$$

With these results, the action of the antiferromagnet takes on the simple form

$$S = \tfrac{1}{2}\int {\rm d}^2x\, I_a[\omega^a_\mu]^2, \tag{4.22}$$

where summation over $a = (1,2,3)$ and $\mu = (1,2)$ is implied. The quantities I_a define the stiffness for twisting the order parameter about each of the principal axes. In this form, the antiferromagnet is the simple generalization of a top from one time dimension to two (space) dimensions.

Equation (4.22) can be regarded as a Landau–Ginzburg action for the long-wavelength modes of the two-dimensional antiferromagnet. However, unlike in neutral U(1) superfluids, the long-wavelength spin fluctuation modes are *strongly interacting* due to the fundamentally non-Abelian nature of rotations. These spin-wave interactions screen the stiffness to zero at long length-scales, thus disordering the antiferromagnet. This phenomenon was first studied in detail by Polyakov [47], who derived the scaling of the spin stiffness for the isotropic (and the uniaxial) limit by successfully removing spin fluctuations of long and longer wavelength, iteratively computing the stiffness associated with the screening induced by each momentum shell. The general anisotropic case was first treated in the context of two-dimensional gravity by Friedan [52]. Independently, Polyakov and Wiegmann [53] have shown its equivalence to the anisotropic Kondo model, whose scaling equations were derived by Anderson and Yuval [54,4]. Most recently Azaria et al. [55] have cast the general results of Friedan [52] into a form more appropriate to the study of magnetism.

4.2. *Polyakov scaling of the spin stiffness*

The basic philosophy behind Polyakov scaling is to divide the spin fluctuations into short- and long-wavelength components, integrating out the fast degrees of freedom while maintaining the spin amplitude fixed; this is a "poor man's scaling" approach to magnetism [54,4]. Before generalizing this prescription to the anisotropic case, we briefly review conventional Polyakov scaling for the uniaxial limit. We then extend this method to *non-collinear* magnets; there, as we shall see, there are two ways in which this decimation procedure can be followed depending on whether one studies the scaling in the body-axis or in the laboratory reference frame.

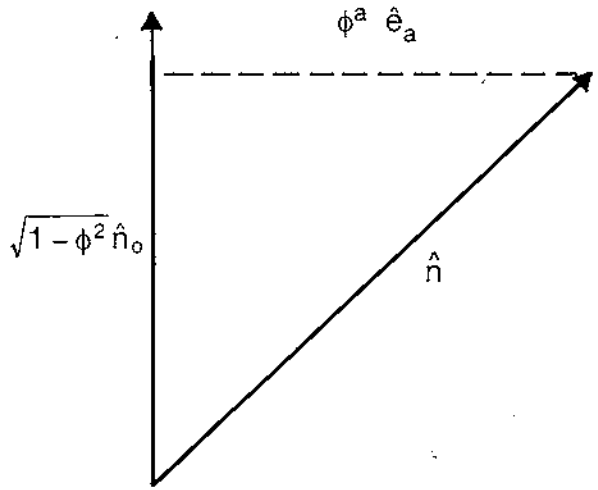

Fig. 9. Diagram representing the short- ($\{\hat{e}_a\}$) and long- ($\hat{n}_0$) wavelength fluctuations in Polyakov's scaling treatment of a fixed-amplitude unit vector.

4.2.1. The O(3) case

For O(3) (collinear) magnets, the general long-wavelength action (4.10) reduces to

$$S = \frac{1}{2g}\int \mathrm{d}^2x\,(\vec{\nabla}\hat{n})^2 \qquad (\hat{n} \equiv \hat{e}_3), \tag{4.23}$$

where $\hat{n} = \hat{n}_A$ is the staggered magnetization on sublattice A and the coupling constant $g = T/(2JS^2) \equiv 2/I$. Following Berezinskii and Blank [56], the local order can be decomposed into short- and long-wavelength degrees of freedom,

$$\hat{n}(x) = \hat{n}_0(x)\sqrt{1-\phi^2(x)} + \sum_{a=1}^{n-1}\phi^a(x)\hat{e}_a(x), \tag{4.24}$$

which is displayed graphically in fig. 9; the $\phi^a(x)$ are the generators of the rotation matrix in eq. (4.5) for the special O(3) case. Here $\hat{n}_0 \equiv \hat{e}_3(0)$ is a *slowly* varying vector that forms an orthonormal basis with the $\hat{e}_a$; the "fast" fluctuations, $\phi^a(x)$, have a bandwidth given by $\tilde{\Lambda} < |q| < \Lambda$, where Λ and $\tilde{\Lambda}$ are the bare and the renormalized momentum cutoffs (to be determined from the scaling), respectively.

Our immediate goal is to rewrite the action (4.23) in terms of these new fast and slow degrees of freedom. We write the slow variation in $\hat{n}_0$ as

$$\nabla_\mu\hat{n}_0 = B^a_\mu\hat{e}_a\,, \tag{4.25}$$

which is a fancy version of

$$\hat{n}_0(\vec{r}+\Delta\vec{r}) = \hat{n}_0(\vec{r}) + (\Delta r_\mu B^a_\mu)\,\hat{e}_a(\vec{r}). \tag{4.26}$$

We are only interested in the "fast" degrees of freedom ϕ to Gaussian order; eqs. (4.24) and (4.25) lead to

$$\nabla_\mu\hat{n} = [\nabla_\mu\phi^a + (1-\tfrac{1}{2}\phi^2)B^a_\mu]\,\hat{e}_a - [\phi^a B^a_\mu]\,\hat{n}_0, \tag{4.27}$$

where we have used the short-hand

$$\nabla_\mu\phi_a \equiv \nabla_\mu\phi^a - A^{ba}_\mu\phi^b. \tag{4.28}$$

We can now write the action (4.23) to quadratic order in the "fast" fluctuations as

$$S = \frac{1}{g}\int \mathrm{d}^2x\,\{(\nabla_\mu\phi^a)^2 + (\nabla_\mu\hat{n})^2 + (\phi^a\phi^b e - \phi^2\delta^{ab})B^a_\mu B^b_\mu + \mathrm{O}(\phi^3)\}. \tag{4.29}$$

In order to get the scaling equation we must evaluate $\phi^a\phi^b$ and ϕ^2 in eq. (4.29); because this is a Gaussian theory we can approximate them by their thermodynamic averages $\langle\phi^a\phi^b\rangle$ and $\langle\phi\rangle$. Assuming that ϕ_a and ϕ_b are independent to Gaussian order, we write

$$\langle\phi_a\phi_b\rangle = \int_{\Lambda-\delta\Lambda\equiv\tilde{\Lambda}}^{\Lambda}\frac{\mathrm{d}^2k}{(2\pi)^2}\,\langle\phi^2_k\rangle\delta^{ab} = \frac{g}{4\pi}\delta^{ab}\frac{\delta\Lambda}{\Lambda}\,, \tag{4.30}$$

where we have used $\langle\phi^2_k\rangle = g/k^2$. Therefore,

$$\langle\phi^2\rangle = (n-1)\langle(\phi^a)^2\rangle = \frac{n-1}{4\pi}\,g\,\frac{\delta\Lambda}{\Lambda} \tag{4.31}$$

and we get the scaling equation

$$\frac{1}{2\tilde{g}} = \frac{1}{2g} + \frac{n-2}{4\pi}\,\mathrm{d}\ln\Lambda, \tag{4.32}$$

where $1/g$ and $1/\tilde{g}$ are the bare and renormalized coupling constants, respectively, and we have used $\delta\Lambda/\Lambda = \mathrm{d}\ln\Lambda$. We can rewrite eq. (4.32) in the conventional scaling form

$$\frac{\partial(1/g)}{\partial \ln \Lambda} = \frac{1}{2\pi}(n-2), \tag{4.33}$$

or as

$$\frac{\partial g}{\partial \ln \Lambda} = -\frac{1}{2\pi}(n-2)g^2. \tag{4.34}$$

Physically eq. (4.33) tells us the scaling of the spin stiffness $1/g$ as a function of n, the number of spin components; it is unrenormalized for xy magnets ($n = 2$) but scales to weak coupling in the Heisenberg case ($n = 3$). We can also obtain the spin correlation length from eq. (4.33); it yields the "running" coupling constant

$$g(\Lambda) = \frac{g_0}{1 + [g_0(n-2)/2\pi]\ln(\tilde{\Lambda}/\Lambda)}. \tag{4.35}$$

At the onset of strong-coupling ($g = 1$) we have the condition

$$\frac{\tilde{\Lambda}}{\Lambda} = \frac{a}{\xi}, \tag{4.36}$$

where a is the lattice cutoff. Equations (4.35) and (4.36) lead to the spin correlation length

$$\xi \sim a \exp\left[\frac{\pi}{n-2}\frac{2JS^2}{T}\right]. \tag{4.37}$$

We see from eq. (4.37) that ξ will always be finite at $T \neq 0$ for the Heisenberg model, thus indicating that here long-wavelength thermal fluctuations will always suppress the formation of disorder. By contrast, the spin correlations are power-law in the xy case; stable defects (vortices) can then interact on long length-scales, leading to the famous defect-binding transition of Kosterlitz and Thouless [57].

4.2.2. The anisotropic scaling equations

We now generalize Polyakov's method to the *anisotropic* antiferromagnet; a complementary approach, using the ϵ-expansion, has been discussed by Azaria et al. [55]. The order parameter for a general *non-collinear* magnet is a rotation matrix; following the general prescription for Polyakov scaling, we would like to express it as the product of two rotation matrices associated with the slow and fast spatial

degrees of freedom. Unlike the O(3) case this renormalization procedure can be implemented in more than one way, depending on whether one wants to measure the stiffness in the lab or in the body-axis frame. For example, if we are interested in the scaling of the stiffness due to a change of boundary conditions we must measure it in the lab frame; then the fast degrees of freedom must be described with reference to the body axes

$$g(x) = g_0(x)h(x). \tag{4.38}$$

Alternatively, we may want to study the stiffness in the body frame; then we must write the order parameter

$$g(x) = h(x)g_0(x). \tag{4.39}$$

The second procedure, i.e. eq. (4.39), results in a simpler pair of scaling equations; heuristically, this is because the body axes provide a more natural reference frame for the low-energy modes that move with the "space-time top". Numerically, however, this stiffness is more difficult to measure. In a classical magnet with intrinsically *uniaxial* spin components there is no well-defined body axis; the first renormalization procedure described above is necessary just to explicitly integrate out the fast gauge rotations about the axis of magnetization.

Here, for the sake of simplicity, we will use the decomposition eq. (4.39); the alternative procedure using eq. (4.38) is described in detail by Azaria et al. [55]. Using the decomposition eq. (4.38), we write the angular velocity matrix (4.16) as

$$\begin{aligned}
\underline{\omega}_\mu &= g_0^{-1}\underline{\alpha}_\mu g_0 + \underline{\Omega}_\mu, \\
\underline{\alpha}_\mu &= h^{-1}\nabla_\mu h, \\
\underline{\Omega}_\mu &= g_0^{-1}\nabla_\mu g_0,
\end{aligned} \tag{4.40}$$

so that the action (4.10) is

$$S = \tfrac{1}{2}\int \mathrm{d}^2x\ \mathrm{Tr}\left[\underline{K}(g_0^{-1}\underline{\alpha}_\mu g_0 + \underline{\Omega}_\mu)^2\right], \tag{4.41}$$

with the slow modes (Ω_μ) acting as an external field on the fast rotations (α_μ). We can identify $-\vec{\Omega}_\mu$ as spin vector potential, similar to that discussed in the last two sections. The spin current is then

$$\begin{aligned}
\vec{J}_\mu &= \frac{\partial F}{\partial \vec{\Omega}_\mu} = \underline{I}\cdot[\langle\vec{j}_\mu\rangle + \vec{\Omega}_\mu], \\
\vec{j}_\mu &= \underline{g}_0^{-1}\underline{\alpha}_\mu\underline{g}_0,
\end{aligned} \tag{4.42}$$

and the stiffness is

$$\frac{\partial^2 F}{\partial \vec{\Omega}_\mu \partial \vec{\Omega}_\nu} = \underline{I}\delta_{\mu\nu} - \langle \vec{j}_\mu \vec{j}_\nu \rangle, \tag{4.43}$$

which is a form very similar to that discussed in section 2. The second term in eq. (4.43) represents the "paramagnetic" current fluctuations; they renormalize the spin stiffness, ultimately driving it to zero at finite-temperatures and generating a finite spin correlation length.

In order to compute the leading (one-loop) renormalization of the stiffnesses by the spin-current fluctuations, we expand the fast fluctuations to Gaussian order

$$\begin{aligned} h &= \mathrm{e}^{\underline{\theta}} = 1 + \underline{\theta} + \tfrac{1}{2}\underline{\theta}^2 + \mathrm{O}(\underline{\theta}^3), \\ \underline{\alpha}_\mu &= \omega_\mu^a \underline{t}_a = \nabla_\mu \underline{\theta} + \tfrac{1}{2}\left[\nabla_\mu \underline{\theta}, \underline{\theta}\right], \\ \alpha_\mu^a &= [\nabla_\mu \vec{\theta} + \tfrac{1}{2}(\nabla_\mu \vec{\theta} \times \theta)]^a, \end{aligned} \tag{4.44}$$

and take the fast fluctuations to lie within a narrow shell in momentum space $q \in [\Lambda, \Lambda + \mathrm{d}\Lambda]$. We assume that g_0 varies slowly on the spatial scale of $1/\Lambda$; for the calculation of the fast fluctuations, it is approximated as a "constant" rotation that commutes with the gradient operator. Formally

$$\begin{aligned} &\underline{\omega}_\mu^* = \omega_\mu^{*a} \underline{t}_a, \\ &\underline{g}_0^{-1} \underline{\alpha}_\mu \underline{g}_0 = \left(\underline{g}_0^{-1} \underline{t}_a \underline{g}_0\right) \alpha_\mu^a = \underline{t}_b \left[\underline{g}_0\right]_{ba} \alpha_\mu^a, \end{aligned} \tag{4.45}$$

so that

$$\vec{\omega}_\mu^* = \underline{g}_0 \vec{\alpha}_\mu . \tag{4.46}$$

Using eq. (4.44) and commuting the rotation matrix through the gradients we find that

$$\vec{\omega}_\mu^* = [\vec{\nabla}_\mu \vec{\theta}' + \tfrac{1}{2}(\vec{\nabla}_\mu \vec{\theta}' \times \theta')]^a, \tag{4.47}$$

where

$$\begin{aligned} \vec{\nabla}_\mu &\equiv \nabla_\mu + g_0 \underline{\Omega}_\mu g_0^{-1}, \\ \vec{\theta}' &= g_0 \theta'. \end{aligned} \tag{4.48}$$

Since $|\vec{\Omega}| \ll \Lambda$, we will neglect the difference between $\vec{\nabla}$ and ∇ in $\vec{\omega}_\mu^*$.

Performing a Fourier transform on the angular variables, we can now express the action to Gaussian order in the fluctuations.

$$\begin{aligned} S[\theta', \Omega] &= \tfrac{1}{2} \int \mathrm{d}^2 x \, \vec{\Omega}_\mu \cdot \underline{I} \cdot \vec{\Omega}_\mu + \sum_{\vec{q}} \vec{\theta}'_{-\vec{q}} \underline{M}_{\vec{q}} \vec{\theta}'_{\vec{q}}, \\ [\underline{M}_{\vec{q}}]_{ab} &= q^2 I_a + \mathrm{i} q_\mu \epsilon_{abc} I_c \Omega_\mu^c . \end{aligned} \tag{4.49}$$

We renormalize the action by integrating over the "fast" degrees of freedom,

$$\exp\left(-S_{[\mathrm{eff}]}[\Omega]\right) = \int \mathrm{d}[\theta'_{\vec{q}}] \, \mathrm{e}^{-S[\theta', \Omega]}, \tag{4.50}$$

to obtain

$$S_{[\mathrm{eff}]}[\Omega] = \tfrac{1}{2} \int \mathrm{d}^2 x \, \vec{\Omega}_\mu \cdot \underline{I} \cdot \vec{\Omega}_\mu + \sum_{\vec{q}} \mathrm{Tr}(\ln M_{\vec{q}}). \tag{4.51}$$

To obtain the renormalized stiffness we next differentiate with respect to Ω:

$$\begin{aligned} \frac{1}{V} \frac{\partial^2 S_{[\mathrm{eff}]}}{\partial \Omega_\mu^a \partial \Omega_\nu^b} &= I_a^* \delta_{ab} \delta^{\mu\nu}, \\ I_a^* &= I_a - \frac{I_a^2}{I_b I_c} \int_{q \in [\Lambda, \Lambda + \mathrm{d}\Lambda]} \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{1}{q^2} \\ &= I_a - \frac{1}{4\pi} \frac{I_a^2}{I_b I_c} |\mathrm{d} \ln \Lambda| \qquad (b \neq c \neq a), \end{aligned} \tag{4.52}$$

thus obtaining the scaling equation

$$\frac{\mathrm{d}g_a}{\mathrm{d} \ln \Lambda} = -\frac{1}{4\pi} g_b g_c \qquad (a \neq b \neq c), \tag{4.53}$$

where $g_a \equiv 1/I_a$; eq. (4.53) describes the renormalization of the (inverse) stiffness by long-wavelength spin-current fluctuations.

Usually, a magnet will have at least one axis of symmetry with two equal stiffnesses, i.e. $g_1 = g_2$ and $g_3 = g_\parallel$. Now, the scaling (4.53) simplifies:

$$\begin{aligned} \frac{\mathrm{d}g_1}{\mathrm{d} \ln \Lambda} &= -\frac{g_1 g_3}{4\pi}, \\ \frac{\mathrm{d}g_3}{\mathrm{d} \ln \Lambda} &= -\frac{(g_1)^2}{4\pi}. \end{aligned} \tag{4.54}$$

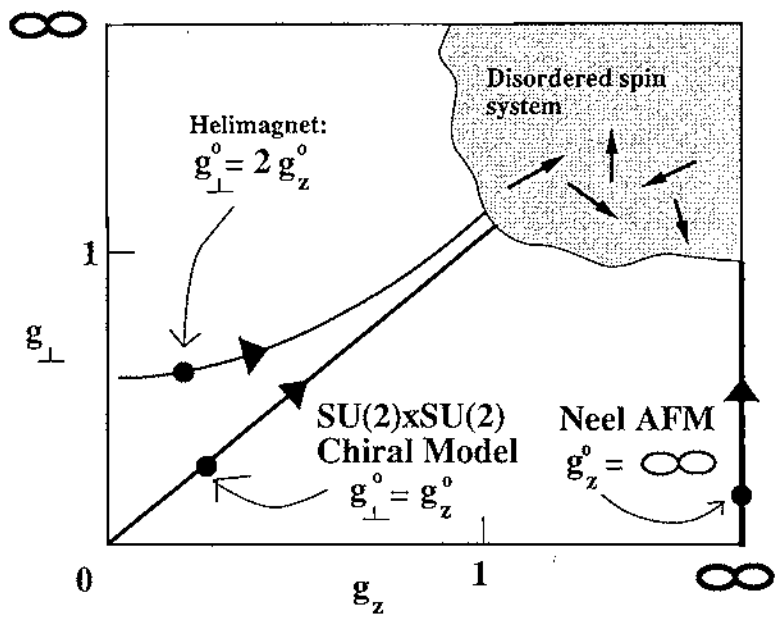

Fig. 10. Scaling trajectories for anisotropic two-dimensional σ-models.

There is even a first integral,

$$(g_3)^2 - (g_1)^2 = C^2, \tag{4.55}$$

that yields the explicit solution

$$g(\Lambda) = C \coth\left[\frac{C}{4\pi}\ln\left(\frac{\Lambda_0}{\Lambda}\right)\right]. \tag{4.56}$$

Equations (4.54) are the Anderson–Yuval scaling equations [54], first derived for the anisotropic Kondo problem. Their scaling trajectories are displayed in fig. 10; here we see that fluctuations reduce the stiffness, thus enhancing the degree of isotropy. At the spin correlation length,

$$\xi = a\,\exp\left[\frac{4\pi}{C}\coth^{-1}\frac{1}{C}\right] \sim a\,e^{4\pi/C} \qquad (a = \Lambda_0^{-1}), \tag{4.57}$$

the system scales to strong coupling ($g_3 \sim$ O(1)); at distances $d > \xi$ the fluctuations act nonperturbatively, and the magnet is disordered. As an aside we note that the Anderson–Yuval form of the scaling equations (4.26) is not entirely coincident; Babujian and Tsvelik [58] have shown that the 2D non-linear σ-model is equivalent to the large-channel limit of the (1+1)-D multichannel Kondo problem, so that the former can be solved exactly by the Bethe Ansatz technique.

The renormalization procedure followed by Azaria et al. [55] produces additional terms in the scaling equations which account for the fluctuations of the body-axis reference frame relative to the fixed coordinates. At first sight, it might seem unusual that there are two possible renormalization procedures; clearly, both must lead to the same results for physical quantities, as can be verified in perturbation theory. Beyond perturbation theory, the general scale-dependent transformation between the body-axis and laboratory reference frames is highly non-linear, and thus the usefulness of each procedure depends on the chosen reference frame in which one wishes to follow the renormalization-group flows.

We note that these equations (4.26) do *not* allow us the study the cross-over from biaxial to uniaxial behavior; in this limit the fluctuations about the symmetry axis are unconstrained and must be factored out from the partition function as a gauge degree of freedom. It is clear, then, that such a passage from O(3) to SO(3) magnetism can only occur via a strong-coupling regime, for small deviations from perfect O(3) behavior act as strong-coupling terms in the action that cannot be gauged away. In the related Kondo problem, where the explicit strong-coupling solution is known via the Bethe Ansatz, such a continuous passage to the O(3) model has been performed by Weigman [59]. However, in the quantum case, almost nothing is known about the transition from O(3) to SO(3); several groups have speculated about a possible intermediary "spin-liquid" regime, though others believe that such a disordered phase will be preempted by a first-order phase transition. This issue will be discussed further in section 5, when we review various "moment-free" phases proposed by the community.

4.3. Order from disorder: a simple example

The conventional scaling picture of magnetism incorporates *all* fluctuation effects into an effective Landau–Ginzburg action for the long-wavelength modes of the system. However, even in cases where such an action is well-defined, high-frequency short-wavelength spin fluctuations can modify its behavior at long distances. The role of such short-wavelength fluctuations is particularly enhanced in frustrated spin systems with large ground-state degeneracies; here Villain [23] has observed that the associated fluctuation-free energy often selects spin configurations that break the underlying lattice symmetry. In the quantum case, these states minimize their zero-point energy, thereby maximizing their number of zero modes. This is somewhat counterintuitive: normally we associate stability with rigidity. In frustrated systems, Villain has turned this standard argument on its head: he has shown that the "most flexible" spin configurations are least affected by the presence of fluctuations, and thus outlast their more "rigid" counterparts in the degenerate ground-state manifold [23].

Villain's original discussion of fluctuation-induced ordering was confined to

Ising magnets, and was subsequently extended to continuous spins by Shender [60] and Henley [61]. In these systems, the presence of a continuous degeneracy in the classical ground state leads to "false" zero modes, that are subsequently lifted by fluctuations [60]. Such "exchange gaps", in this case due to zero-point fluctuations, have been observed in dynamical neutron studies on the spin-$\frac{5}{2}$ garnet $Fe_2Ca_3(GeO_4)_3$; here the measured dispersion curves are reproduced by a simple frustrated model, and the observed quantum gaps agree with a higher-order spin-wave calculation [62]. Villain's "order from disorder" provides a clear example of how the interplay between fluctuations and competing interactions can, in principle, lead to new forms of spin order. Since the fluctuation-selection of spin states plays a central role in the "quest" for exotic magnetism, we now examine this curious phenomenon in the context of a simple example.

We turn to the two-dimensional frustrated square Heisenberg Hamiltonian

$$H = J_1 \sum_{ij} \vec{S}_i \cdot \vec{S}_j + J_2 \sum_{ik} \vec{S}_i \vec{S}_k \,, \tag{4.58}$$

with nearest- and next-nearest-neighbor couplings J_1 and J_2,respectively; in the limit $\eta = J_1/(2J_2) \ll 1$ this simple model has a ground state with a continuous global degeneracy [63]. For $\eta > 1$ the classical ground state has conventional Néel order, but when $\eta < 1$ the two sublattices become decoupled and can have *arbitrary* angular orientation with respect to one another (see fig. 11). Classically this ground state ($\eta > 1$) has energy

$$E = S^2\{-J_2 + 2J_1 \cos\theta + J_2 - 2J_1 \cos\theta\} = S^2\{-2J_2\}. \tag{4.59}$$

The spin-wave spectrum, which is sensitive to short-wavelength fluctuations, suggests that "order from disorder" effects could occur; in the limit $\eta \ll 1$ it is given by [25]

$$\omega(\vec{q},\theta)^2 = (4SJ_2)^2\{[1 + \eta(\alpha \cos q_x + \beta \cos q_y)]^2 - [\cos q_x \cos q_y + \eta(\alpha \cos q_y + \beta \cos q_x)]^2\}, \tag{4.60}$$

where $(\alpha, \beta) = (\cos^2 \frac{1}{2}\theta, \sin^2 \frac{1}{2}\theta)$ and θ is the angle between the two-sublattices (see fig. 12). The angle-dependent part of the classical fluctuation free energy, $\delta F(\theta) = F(\theta) - F(0)$ can be estimated to leading order by incorporating the zero-temperature dispersion (4.60) into

$$F(\theta) = T \sum_{\vec{q}} \ln \frac{\omega(\vec{q},\theta)}{2T}, \tag{4.61}$$

which, upon integration, yields

$$\delta F(T,\theta) \sim -E(T)(1 + \cos^2\theta) \tag{4.62}$$

for $\eta \ll 1$, where $E(T) = 0.636\,\eta^2 T$.

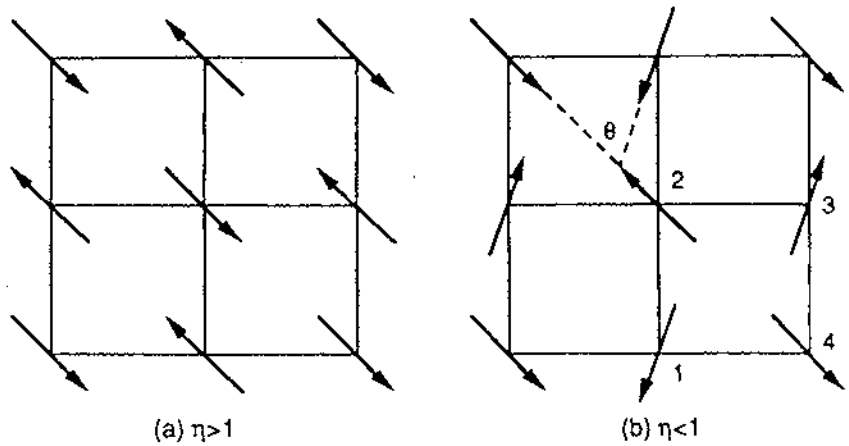

Fig. 11. The classical ground states for the 2D frustrated square Heisenberg antiferromagnet with (a) $\eta > 1$ and (b) $\eta < 1$, where $\eta \equiv J_2/J_1$ is the ratio of the nearest- and next-nearest-neighbor couplings; note that in (b) the ground state has a *continous* degeneracy associated with the angular independence of the two sublattices (reprinted from Ritchey [13]).

The fluctuation free energy (4.62) clearly favors spin configurations where $\cos\theta = \pm 1$ (fig. 12), thereby retaining a *discrete* two-fold degeneracy in the ground-state manifold. It selects states that break Z_4 lattice symmetry; the spins are ferromagnetically aligned ($\omega \sim k^2$) in one lattice direction, thereby *maximizing* the coupling between the two sublattices and minimizing the overall dispersion. This behavior is easier to understand in the analogous ferromagnetic J_1, J_2 model ($J_1, J_2 < 0$). Here, zero-point motion is completely eliminated when the two sublattices are parallel, forming a uniform ferromagnet. In the antiferromagnetic case of interest, fluctuations can never be eliminated, but they are minimized in the configuration that is *maximally ferromagnetic*. From a more technical standpoint, the coupling between two antiferromagnetic sublattices enters as off-diagonal matrix elements in a standard spin-wave calculation; thus maximum coupling leads to a minimization of the dispersion, and thus the free energy. Fluctuations favor configurations with $\sigma = \pm 1$, where

$$\sigma = (\vec{S}_1 - \vec{S}_3) \cdot (\vec{S}_2 - \vec{S}_4)/2S^2 \tag{4.63}$$

is defined in terms of the four spins on a square plaquette.

4.3.1. Modified scaling arguments

The fluctuation free energy (4.62) suggests that a *quadrupolar* coupling term

$$I_c = -\frac{1}{2g} \int d^2x \, \frac{(\hat{n}_1 \cdot \hat{n}_2)^2}{l^2(g)} \tag{4.64}$$

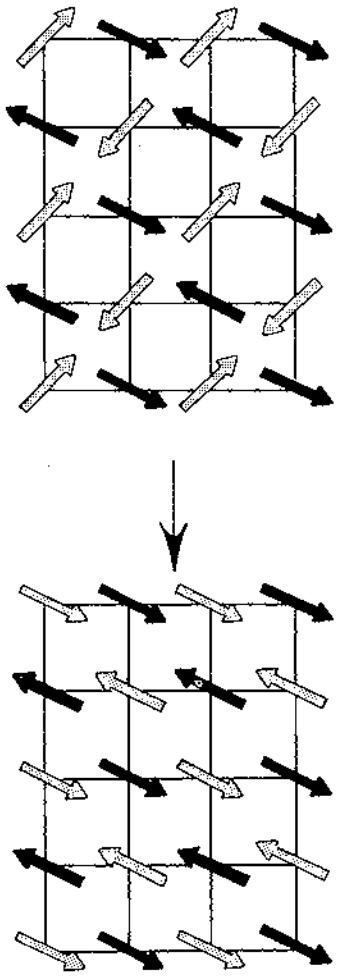

Fig. 12. Fluctuation selection of ground states in the 2D frustrated square Heisenberg antiferromagnet. Note that although fluctuations lift the continuous degeneracy of the classical manifold, a discrete Z_2 degeneracy remains (reprinted from Ritchey [13]).

should be added to the classical action

$$I = \frac{1}{2g}\int \mathrm{d}^2x \left\{ \sum_{i=1,2} (\nabla \hat{n}_i)^2 + 2\eta(\nabla_x \hat{n}_1 \cdot \nabla_x \hat{n}_2 - \nabla_y \hat{n}_1 \cdot \nabla_y \hat{n}_2) \right\} \tag{4.65}$$

derived from a gradient expansion in the usual fashion. The quadrupolar coupling

term (4.64) selects configurations with $\sigma = \pm 1$, and

$$l = a\sqrt{\frac{T}{E(T)g(l)}} \tag{4.66}$$

is the wall thickness between $\sigma = +1$ and $\sigma = -1$ configurations. Because this discrete variable is stabilized by short-wavelength fluctuations it should be robust to infrared divergences, and thus should survive to finite temperatures in the *absence* of true long-range antiferromagnetic order. In particular, we expect that the soft Ising order parameter σ will be finite if the wall thickness is less than the spin correlation length; we expect an order–disorder transition at $l(T) = \xi(T)$. At this temperature the energy barrier separating x-collinear ($\sigma = 1$) and y-collinear ($\sigma = -1$) states is of height

$$W(T) = \frac{E(T)}{a^2}\int_{r<\xi/2} \mathrm{d}^2x = E(T)\frac{\xi(T)}{a}. \tag{4.67}$$

Roughly speaking the order-from-disorder transition temperature is determined by the condition

$$W(T_{\mathrm{i}}) = T_{\mathrm{i}}, \tag{4.68}$$

which yields

$$T_{\mathrm{i}} = \frac{2\pi J_2 S^2}{\ln[T_{\mathrm{i}}/E(T_{\mathrm{i}})]}, \tag{4.69}$$

where we have estimated the spin correlation length by

$$\xi \sim a \exp \frac{2\pi J_2 S^2}{T_{\mathrm{i}}}, \tag{4.70}$$

a reasonable assumption in the large-J_2 limit.

A Polyakov scaling treatment of the full action (4.65) indicates that there are two distinct scaling regimes in this model [25]. At high temperatures the wall thickness (4.66) is much longer than the spin correlation length ξ; thus walls between different collinear regions are *not* defined, and the two sublattices are unlocked. However, when l is shorter than ξ the quadrupolar coupling term (4.64) becomes larger than eq. (4.65); eq. (4.64) results in a "quantum exchange gap" $\Delta \sim (c/l_{T=0})$ for out-of-phase fluctuations, thereby locking the two sublattices (see fig. 13) and stabilizing the soft Ising order parameter σ even in the absence of a sublattice magnetization. The Ising transition is found from the condition $l \sim \xi$.

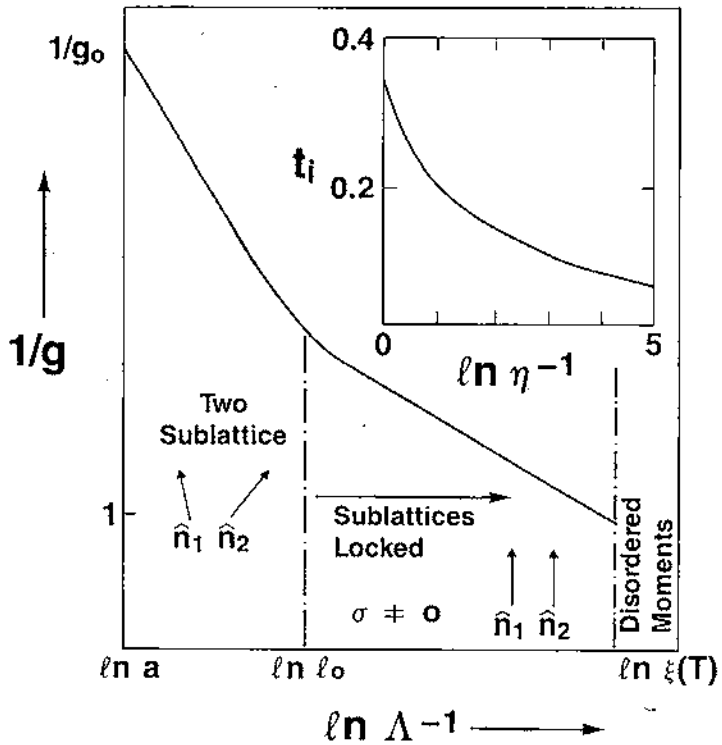

Fig. 13. The scaling behavior of the spin stiffness as a function of length scale in the frustrated square Heisenberg model in the limit of large J_2. Inset: the Ising phase transition as a function of frustration.

and in the small-η limit the scaling prediction is in good agreement with the estimated value (4.69) above. Heuristically $T_{\rm I}^{*}$ can be described as a "spin-binding" transition temperature; for $T < T_{\rm I}^{*}$ the spin stiffness is $1/g^{*} = 2J(S^{*})^2/T$, where $S^{*} = 2S$. Consequentially we expect that, in contrast with results for the O(3) σ-model [41,37], our conclusions should be stable against collective topological effects. Since the size of the point defect $l_{\rm d}$ is much larger than that of an Ising wall l, the effective spin in the Berry phase calculation must be $S^{*} = 2S$, which is always an integer. When S^{*} is even there will be no effects, but when S^{*} is odd, the collective tunneling between different hedgehog configurations will only serve to reinforce the two-fold degeneracy already driven by the fluctuations. Since S^{*} is never half-integer, it appears that fluctuation-stabilized "order from disorder" suppresses the topologically generated four-fold degenerate spin Peierls states [25].

4.3.2. The quantum fluids perspective

Within the quantum fluids picture, the basic character of this "spin-binding" transition is determined by the constraint equation which self-consistently determines the temperature-dependent quantum exchange gap; physically this phenomenon is a result of fluctuation effects on the Onsager reaction field [38]. To see this, we recall the pairing equations and the constraint

$$\begin{aligned} h_{\vec{q}} &= \int_{\vec{q}'} \mathcal{J}^{+}_{\vec{q}\vec{q}'}\alpha_{\vec{q}'}\,, \\ \Delta_{\vec{q}} &= \int_{\vec{q}'} \mathcal{J}^{-}_{\vec{q}\vec{q}'}\eta_{\vec{q}'}\,, \\ S + \tfrac{1}{2} &= \int_{\vec{q}} \alpha_{\vec{q}}\,, \end{aligned} \tag{4.71}$$

where $\int_{\vec{q}} \equiv \int {\rm d}^2q/(2\pi)^2$ and

$$(2\alpha_{\vec{q}},\ 2\eta_{\vec{q}}) = (\langle D^{\dagger}_{\vec{q}}\rangle, \langle B^{(0)\,\dagger}_{\vec{q}}\rangle) = [\coth(\beta\omega_{\vec{q}}/2)/\omega_{\vec{q}}]\,(\tilde{h}_{\vec{q}}, \Delta_{\vec{q}}), \tag{4.72}$$

with $\omega^2_{\vec{q}} = \tilde{h}^2_{\vec{q}} - \Delta^2_{\vec{q}}$ and $\tilde{h}_{\vec{q}} = h_{\vec{q}} - \lambda$. We consider the low-temperature collinear state with ferromagnetic correlations along the x-axis and Goldstone modes at $\vec{q} = 0, \vec{Q}$, with $\vec{Q} = (0, \pi)$; for this case the pairing fields are

$$\begin{aligned} \mathcal{J}^{(+)}_{\vec{q}\vec{q}'} &= 4J_2\epsilon c_x c'_x\,, \\ \mathcal{J}^{(-)}_{\vec{q}\vec{q}'} &= 4J_2[c_x c_y c_{x'} c_{y'} + \epsilon c_y c'_y], \end{aligned} \tag{4.73}$$

with $\epsilon = J_1/(2J_2) \ll 1$.

We would now like to determine the fluctuation-induced quantum exchange gap at $\vec{Q}^{*} = (\pi, 0)$ and (π, π). After the first iteration of the pairing equations (4.71) it is

$$\begin{aligned} (\Delta_1)^2 &= 2\{\tilde{h}_{\vec{Q}^*}\delta\tilde{h}_{\vec{Q}^*} - \delta\Delta_{\vec{Q}^*}\delta\Delta_{\vec{Q}^*}\} \\ &= 2\tilde{h}_{\vec{Q}^*}\{(\delta h_{\vec{Q}^*} - \delta h_0) + (\delta\Delta_0 + \delta\Delta_{\vec{Q}^*})\} = c^2/l^2_{T=0}\,, \end{aligned} \tag{4.74}$$

where we have used $h_{\vec{Q}^*} = -\Delta_{Q^*}$ and $\delta\lambda = \delta h_0 - \delta\Delta_0$; here $\delta\Delta_{\vec{q}}$ and $\delta h_{\vec{q}}$ are the deviations from the zero-temperature, infinite-spin values of these quantities. Then we have

$$(\Delta_1)^2 = 2\tilde{h}_{\vec{Q}^*}\left\{\sum_{q}(\mathcal{J}^{(+)}_{\vec{Q}^*\vec{q}} - \mathcal{J}^{(+)}_{0\vec{q}})\alpha_{\vec{q}} + (\mathcal{J}^{(-)}_{\vec{Q}^*\vec{q}} + \mathcal{J}^{(-)}_{0\vec{q}})\eta_{\vec{q}}\right\}. \tag{4.75}$$

Equations (4.72), (4.73), (4.75) and the dispersion relation

$$\omega_{\vec{q}}^2 = S^2[J(\vec{q}) - J(\vec{Q})]\{\tfrac{1}{2}[J(\vec{q}+\vec{Q}) + J(\vec{q}-\vec{Q})] - J(\vec{Q})\}$$
$$= (4J_2S)^2[(1+\epsilon c_x)^2 - (c_x c_y + \epsilon c_y)^2] \tag{4.76}$$

lead to

$$(\Delta_1)^2 = 32\epsilon(1-\epsilon)S(J_2)^2 \int \frac{\mathrm{d}^2q}{(2\pi)^2}\,\phi(\vec{q}), \tag{4.77}$$

where

$$\phi(\vec{q}) = \frac{(c_y^2 - c_x^2)\epsilon + c_x(c_y^2-1)}{\sqrt{(1+\epsilon c_x)^2 - (c_x c_y + \epsilon c_y)^2}} + \mathrm{O}(1), \tag{4.78}$$

a result identical with the $1/S^2$ spin-wave result of Shender [60]. In the limit of small ϵ we can expand eq. (4.78):

$$(\Delta_1)^2 = 2(S+\tfrac{1}{2})(J_1)^2 \int \frac{\mathrm{d}^2q}{\pi^2}\,\frac{1/2(c_x^2+c_y^2)(1+c_x^2c_y^2) - 2c_x^2c_y^2)}{[1-(c_xc_y)^2]^{3/2}}$$
$$= 4.16\,(S+\tfrac{1}{2})(J_1)^2, \tag{4.79}$$

to retrieve the expression found from a modified scaling analysis [25].

We thus expect spin fluctuations to accumulate in the vicinity these quantum exchange gaps and the true Goldstone modes in the large-S limit. If we approximate the spectrum near these special points by

$$\omega_{\vec{q}}^2 = \begin{cases} c^2(\vec{q}-\vec{Q}_i)^2 + \Delta_0^2 & (\vec{Q}_i = 0, \vec{Q}), \\ c^2(\vec{q}-\vec{Q}_i)^2 + \Delta_0^2 + \Delta_1^2 & (\vec{Q}_i = \vec{Q}^*, (\pi,\pi)), \end{cases} \tag{4.80}$$

where

$$\Delta_0^2 = c^2/\xi(T)^2 = \tilde{h}(\vec{q}=0)^2 - \Delta(\vec{q}=0)^2,$$
$$\Delta_1^2 = \tilde{h}_{\vec{Q}^*}\delta\tilde{h}_{\vec{Q}^*} - \delta\Delta_{\vec{Q}^*}\delta\Delta_{\vec{Q}^*} = c^2/l_0^2, \tag{4.81}$$

then in the large-S limit the finite-temperature constraint equation is

$$S + \frac{1}{2} = \frac{1}{8\pi J_2 S}\left[\int_{\Delta_0}^{c/a} + \int_{\sqrt{\Delta_0^2+\Delta_1^2}}^{c/a}\right] \mathrm{d}x\,\coth[\tfrac{1}{2}\beta x], \tag{4.82}$$

with $x = c|\vec{q} - \vec{Q}_i|$; here a cutoff has been imposed on the momentum integrals in the vicinity of the Goldstone modes and quantum exchange gaps. At high temperatures, where $\Delta_1 = 0$, the spin correlation length is

$$\xi = a\exp\frac{2\pi}{g}, \tag{4.83}$$

with coupling constant

$$\frac{1}{g} = \frac{J_2S^2}{T} + \frac{1}{2\pi}\ln\frac{2Ta}{c}. \tag{4.84}$$

At low temperatures, defined by $\xi \sim l_0$, the out-of-phase "phason" modes are frozen out, locking the two Néel sublattices on length scales greater than l_0. Now the spin fluctuations are concentrated solely in the neighborhood of the Goldstone modes, effectively doubling the spin S of the system; the spin correlation length is

$$\xi(T) = l_0\exp\frac{2\pi}{g^*}, \tag{4.85}$$

where the renormalized coupling constant is

$$\frac{2\pi}{g^*} = 4\pi\left(\frac{1}{g} - \frac{1}{2\pi}\ln\frac{l_0}{a}\right). \tag{4.86}$$

The "spin binding" just described will be accompanied by a lattice symmetry breaking transition; within the quantum fluids framework, this will occur when the pairing

$$\Delta_{\vec{q}} = 2\tilde{\Delta}_1 c_y + 4\tilde{\Delta}_2 c_x c_y \tag{4.87}$$

is anisotropic due to the development of a finite $\tilde{\Delta}_1$; note that eq. (4.87) is a general expression for the Bose pairing field. The coefficient of c_y in the general pairing expression (4.71) for $\Delta_{\vec{q}}$ (where we have put in the specific pairing fields (4.73)) is

$$\tilde{\Delta}_1 = J_1\int_{\vec{q}}\frac{c_y\tilde{\Delta}_{\vec{q}}}{2\omega_{\vec{q}}}\coth(\tfrac{1}{2}\beta\omega_{\vec{q}}). \tag{4.88}$$

Writing $\Delta_{\vec{q}}$ out in full and evaluating eq. (4.88) in the limit $\tilde{\Delta}_1 \to 0$ (i.e. $T \to T_c$) we obtain

$$\frac{1}{J_1} = \int_{\vec{q}}\left\{\frac{c_y^2}{\omega_{\vec{q}}}\coth(\tfrac{1}{2}\beta_i\omega_{\vec{q}}) + 2\tilde{\Delta}_2 c_x c_y^2\frac{\partial}{\partial\tilde{\Delta}_1}\left(\frac{\coth(\frac{1}{2}\beta_i\omega_{\vec{q}})}{\omega_{\vec{q}}}\right)\right\}, \tag{4.89}$$

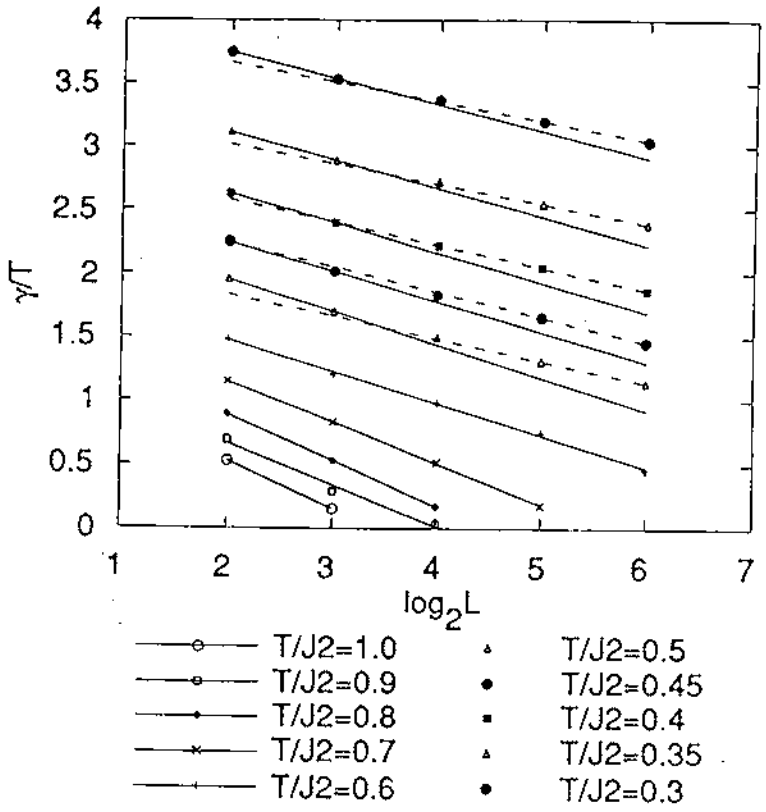

Fig. 14. The spin stiffness γ/T of the J_1–J_2 antiferromagnet with $\eta = J_1/(2J_2) = \frac{1}{40}$ determined by Ritchey [13] from Monte Carlo simulations; γ/T is measured by probing the local spin-current fluctuations.

where the differential is to be evaluated at $\tilde{\Delta}_2 = 0$, and $T_{\rm f} = (\beta_{\rm f})^{-1}$. For small J_1 the integral in eq. (4.89) is dominated by the second term, and we find that

$$T_{\rm f} = \frac{4\pi J_2 S^2}{\ln(\pi T_{\rm f}/J_1 S^2)}, \tag{4.90}$$

which is very similar to that found from a modified scaling treatment; we note that eq. (4.89) provides a more quantitative estimate of the Ising transition temperature than the qualitative condition $l_0(T_{\rm f}) = \xi(T_{\rm f})$ used before.

Perhaps the strongest evidence for a fluctuation-driven lattice symmetry breaking transition comes from numerical measurements of the spin stiffness; exploiting a practical consequence of the quantum fluids approach, Ritchey [13] has probed its behavior on different length scales by monitoring the local spin-current fluctuations. The spin stiffness of the *two* sublattices (i.e. $\gamma/T = 2/g$) is measured; the scaling equations predict that for $T > T_{\rm f}$ (or alternatively for $L < l$ and $T < T_{\rm f}$, where L is the system size and l the wall thickness)

$$\frac{{\rm d}(\gamma/T)}{{\rm d}\ln L} = -\frac{1}{\pi}, \tag{4.91}$$

whereas for $T < T_{\rm f}$ and systems sizes $L > l$

$$\frac{{\rm d}(\gamma/T)}{{\rm d}\ln L} = -\frac{1}{2\pi}, \tag{4.92}$$

when a gap develops in the "out-of-phase" phason mode, the two lattices become locked ("spin binding"), and the effective spin stiffness is doubled. Figure 14 shows Ritchey's results for $\eta = \frac{1}{40}$; there is certainly a crossover in the scaling behavior of γ/T at $L = l \sim 8$ for $T < T_{\rm f} \sim 0.6J_2$. The estimated value of the slope at short length scales and low temperatures in fig. 14, 0.30, is in good agreement with $1/\pi \sim 0.32$ predicted in eq. (4.91); however, the slope at large system sizes, 0.23, is larger than the $1/(2\pi) \sim 0.16$ in eq. (4.92), possibly indicating the break-down of scaling on very long length scales. We note that neither of the analytic approaches described in this section can probe the model at distances significantly longer than the spin correlation length; thus at this point the true nature of the spin correlation length; thus at this point the true nature of the spin binding, i.e. transition or crossover, is still an open question.

5. A quantum zoo of exotic spin order

The fluids approach to quantum antiferromagnetism suggests many novel ground states, thus reemphasizing that these pure spin systems are an "economy" class of the strongly correlated electron problem. What are these various exotic phases, and how and when are they realized? Of course, highly correlated electron fluids, even in the absence of charge fluctuations, are generally very complicated; a mean-field perspective, though far from exact, does give us a flavor for the possibilities. What is the best way to lower the collective energy of a spin system? Conservatively the spins, viewed as particles, form a state associated with a highly *symmetric* wavefunction; this is the traditional antiferromagnet, a "spin superfluid". Can these spin bosons form a pair condensate in the absence of a moment? In the presence of very strong fluctuations, a fermionic representation of the spins may be more appropriate; this is the original RVB perspective first discussed by Anderson [12] and later extended by Baskaran, Zou, and Anderson (BZA) [64]. In two-dimensions particles are not restricted to fermion or boson statistics [65]; is there a spin analogue of