ANALYSIS OF A TRACE CLASS STEKLOFF EIGENVALUE PROBLEM ARISING IN INVERSE SCATTERING

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Abstract.
The Stekloff eigenvalue problem has been a recent problem of interest due to its potential use in nondestructive testing of materials, but it suffers from two shortcomings that are common among this class of eigenvalue methods: it is not known in general if Stekloff eigenvalues exist in the case of complex coefficients, and there exists no mechanism in which to increase their sensitivity to changes in the material properties of the medium under consideration. We present a variation of the Stekloff eigenvalue problem which overcomes both of these issues by the introduction of a certain operator in the boundary condition, and we provide numerical examples to examine the practical consequences of this variation. We discuss this idea in the broader context of what might be called tailored eigenvalue methods, in which the eigenvalue problem is designed to ensure favorable theoretical or practical results.

Key words. inverse scattering, nondestructive testing, non-selfadjoint eigenvalue problems, Laplace-Beltrami operator

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1. Introduction.

A recent subject of interest has been the use of eigenvalues to determine information about the material properties of a scattering medium, and in particular we consider the use of eigenvalues as a potential *target signature*, i.e. a set of numbers which corresponds to a material and may be computed from measured scattering data. An effective target signature allows for a comparison of a potentially damaged material with its undamaged counterpart and consequently serves as an important tool in the nondestructive testing of materials. The theory of transmission eigenvalues is an early example that continues to be an active area of research, and we refer to [7] for a detailed treatment of this topic. However, it has been observed that transmission eigenvalues exhibit some practical shortcomings; in particular, detecting transmission eigenvalues requires the collection of multifrequency data in a predetermined interval. Moreover, only real transmission eigenvalues may be detected from measured scattering data, and for an absorbing inhomogeneous medium it can be shown that no real transmission eigenvalues exist. Both of these issues arise from the observation that transmission eigenvalues are intimately related to the frequency of the interrogating wave, which is a physical parameter and consequently presents physical limitations. From this point of view, a useful alternative is to consider an eigenvalue problem in which the eigenparameter of interest is entirely artificial in nature, and this approach has been recently considered in [3, 8, 9, 10, 11, 12, 13].

Each member of this new class of eigenvalue problems arises by choosing an auxiliary problem that corresponds to mathematically changing the background of the physical scattering problem, and an important note is that the auxiliary problem need not have any relationship to the physical problem. As a result we have a high degree of flexibility in generating these eigenvalue problems. We may leverage this freedom in order to develop what might be called *tailored eigenvalue methods* in which the auxiliary problem is designed to achieve a certain outcome, such as improved

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theoretical results or practical application. An example of this approach arose in [9], in which the authors tailored the auxiliary problem to achieve desirable solvability results for the electromagnetic version of the Stekloff eigenvalue problem. We also mention [4] and [5], in which the authors considered a modification of the transmission eigenvalue problem. A key difference from our present ideas is that the eigenparameter was still related to the physical parameter of frequency. However, the modified spectrum was able to provide improved information on the material coefficients, and consequently this work still falls under the theme of tailored eigenvalue methods.

The main goal of this paper is to present and study a type of auxiliary problem which addresses two pervasive difficulties in this new class of eigenvalue methods. First, it is difficult to establish the existence of eigenvalues when the resulting eigenvalue problem is not self-adjoint (often corresponding to an absorbing material), except in instances when the boundary of the domain and the coefficients are infinitely smooth (cf. [8, 12]). Second, the eigenvalues do not always display sufficient sensitivity to changes in the material to which they correspond, which leads to difficulties in their potential application as a target signature. The general remedy is to include a tuning parameter in the auxiliary problem, as in [3, 10, 12, 13], but at present it has only been available for one type of auxiliary problem. Our solution is a slight modification of previously considered auxiliary problems, and we present it for a specific choice leading to a variation of the Stekloff eigenvalue problem for an inhomogeneous medium. We will see that this generalized Stekloff problem is trace class in the sense that its corresponding Robin-to-Dirichlet map is a trace class operator. This observation will play a vital role in our existence result.

The outline of our paper is as follows. In Section 2 we will briefly introduce the physical scattering problem we will consider, and we will provide a mathematical foundation of the general approach we have discussed up to this point. We will conclude this section with an overview of the standard Stekloff eigenvalue problem and mention the difficulties we will overcome in this particular case. In Section 3 we will introduce our modification of this problem, which will first require us to discuss the properties of the Laplace-Beltrami operator and its spectrum. We will study the so-called δ-Stekloff eigenvalue problem in Section 4 and establish our main result that infinitely many eigenvalues exist, even for an absorbing medium. We will also provide the necessary details to show that δ-Stekloff eigenvalues may be computed from far field data. In Section 5 we will investigate the stability of the δ-Stekloff eigenvalues induced by 1) changes in the material properties, and 2) changes in the parameter δ that appears in the auxiliary problem. We prove stability of an associated solution operator in both cases. In Section 6 we present a series of numerical examples in which we investigate the sensitivity of δ-Stekloff eigenvalues to changes in the material properties and verify that they may be detected from far field data. Finally, we conclude in Section 7 with some remarks on the extension of this approach to other types of scattering problems.

2. The physical scattering problem. We now introduce the physical scattering problem that we will consider for most of our discussion. We consider a function $n \in L^\infty(\mathbb{R}^d)$ $(d = 2, 3)$ representing the refractive index of an inhomogeneous medium, and we assume that the contrast $1 - n$ is supported in a bounded set $\overline{D}$, where D is a Lipschitz domain with connected complement $\mathbb{R}^d \setminus \overline{D}$. We also assume that $\mathrm{Re}(n) \geq n_0 > 0$ and $\mathrm{Im}(n) \geq 0$ a.e. in D. We consider scattering by this inhomogeneous medium of an incident field u^i that satisfies the Helmholtz equation

$$\Delta u^i + k^2 u^i = 0 \text{ in } \mathbb{R}^d$$

for a fixed wave number $k > 0$, and we seek a scattered field $u^s \in H^1_{\text{loc}}(\mathbb{R}^d \setminus \overline{D})$ and a total field $u \in H^1(D)$ such that

$$\Delta u^s + k^2 u^s = 0 \text{ in } \mathbb{R}^d \setminus \overline{D}, \tag{2.1a}$$

$$\Delta u + k^2 n u = 0 \text{ in } D, \tag{2.1b}$$

$$u - u^s = u^i \text{ on } \partial D, \tag{2.1c}$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial u^s}{\partial \nu} = \frac{\partial u^i}{\partial \nu} \text{ on } \partial D, \tag{2.1d}$$

$$\lim_{r\to\infty} r^{(d-1)/2}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0. \tag{2.1e}$$

We assume that the Sommerfeld radiation condition (2.1e) holds uniformly in all directions, and it follows that (2.1a)–(2.1e) is well-posed [7].

The scattered field u^s has the asymptotic form of an outgoing spherical wave with a certain amplitude, and for a plane wave incident field $u^i(x) = e^{ikx\cdot\hat{d}}$ with direction $\hat{d} \in \mathbb{S}^{d-1}$ we write this asymptotic formula as

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}}\left(u_\infty(\hat{x},\hat{d}) + O\left(\frac{1}{|x|^{(d+1)/2}}\right)\right) \text{ as } |x| \to \infty.$$

The function $u_\infty(\hat{x},\hat{d})$ is called the *far field pattern*, and we refer to $\hat{x}$ and $\hat{d}$ as the observation direction and incident direction, respectively. When considering the inverse scattering problem, the measurements of this function at various observation and incident directions serve as our data. A central tool in this analysis is the *far field operator* $F : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ defined by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^{d-1}} u_\infty(\hat{x},\hat{d})g(\hat{d})ds(\hat{d}),\ \hat{x} \in \mathbb{S}^{d-1}.$$

As we mentioned in the introduction, our intended application of eigenvalue methods is to detect changes in the refractive index n from comparing the measured far field pattern to that of a reference medium, and the quality of this comparison will depend on our choice of eigenvalue problem. In order to generate such an eigenvalue problem, we select an auxiliary scattering problem depending on a parameter λ that yields a far field pattern $u_{\lambda,\infty}(\hat{x},\hat{d})$, and we define the corresponding *auxiliary far field operator* $F_\lambda : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ by

$$(F_\lambda g)(\hat{x}) := \int_{\mathbb{S}^{d-1}} u_{\lambda,\infty}(\hat{x},\hat{d})g(\hat{d})ds(\hat{d}),\ \hat{x} \in \mathbb{S}^{d-1}.$$

We remark that the auxiliary far field pattern $u_{\lambda,\infty}(\hat{x},\hat{d})$ need not have any relationship to the physical scattering problem (2.1a)–(2.1e), with the exception that the scattered field which gives rise to $u_{\lambda,\infty}(\hat{x},\hat{d})$ must satisfy the Helmholtz equation with the same wave number k in the exterior of some ball centered at the origin. The resulting eigenvalue problem will arise from a study of the *modified far field operator* given by $\mathcal{F}_\lambda := F - F_\lambda$, which may be expressed as

$$(\mathcal{F}_\lambda g)(\hat{x}) := \int_{\mathbb{S}^{d-1}} \Big[u_\infty(\hat{x},\hat{d}) - u_{\lambda,\infty}(\hat{x},\hat{d}) \Big] g(\hat{d}) ds(\hat{d}), \ \hat{x} \in \mathbb{S}^{d-1}.$$

In [8] the auxiliary problem was chosen to be the exterior impedance problem of finding an auxiliary scattered field $u^s_\lambda \in H^1_{\text{loc}}(\mathbb{R}^d \setminus \overline{B})$ satisfying

$$\Delta u^s_\lambda + k^2 u^s_\lambda = 0 \text{ in } \mathbb{R}^d \setminus \overline{B}, \tag{2.2a}$$

$$\frac{\partial u^s_\lambda}{\partial \nu} + \lambda u^s_\lambda = -\frac{\partial u^i}{\partial \nu} - \lambda u^i \text{ on } \partial B, \tag{2.2b}$$

$$\lim_{r\to\infty} r^{(d-1)/2} \left(\frac{\partial u^s_\lambda}{\partial r} - iku^s_\lambda \right) = 0, \tag{2.2c}$$

which is well-posed whenever $\text{Im}(\lambda) \geq 0$ [8]. Here B is chosen to be a bounded Lipschitz domain containing D. In this case the modified far field operator $\mathcal{F}_\lambda$ is injective with dense range provided that there exists no nontrivial solution $w \in H^1(B)$ of

$$\Delta w + k^2 n w = 0 \text{ in } B, \tag{2.3a}$$

$$\frac{\partial w}{\partial \nu} + \lambda w = 0 \text{ on } \partial B. \tag{2.3b}$$

A value of λ for which (2.3a)–(2.3b) admits a nontrivial solution is called a *Stekloff eigenvalue*, and the properties of this class of eigenvalues in the present context have been extensively studied in [3] and [8]. However, many interesting questions remain, and here we address two of them. First, it is not known if Stekloff eigenvalues exist in the general case of complex-valued n; the existence results in [8] require that $n \in C^\infty(\overline{B})$ and that B is a smooth domain. As B may be freely chosen (subject to the requirement that $D \subseteq B$), the second condition is not difficult to satisfy, but it severely restricts the admissable functions n. Second, the aim is to infer changes in n from shifts in the Stekloff eigenvalues, but the eigenvalues may not respond in a significant fashion to certain changes in n and there is no manner in which to tune their sensitivity. This point has been addressed in [3], [10], and [12] by introducing a new auxiliary problem that includes such a tuning parameter, but the analysis and implementation become more involved. In the next section we introduce a slight modification of the boundary condition in (2.2a)–(2.2c) which resolves both of these issues.

3. The δ-Stekloff eigenvalue problem. We let B be a smooth domain in $\mathbb{R}^d$ ($d = 2, 3$) with connected boundary ∂B and connected complement $\mathbb{R}^d \setminus \overline{B}$, and we note that ∂B is a smooth closed surface of dimension $d - 1$. Before we can define the main operator of interest, we briefly recall the Laplace-Beltrami operator on ∂B, denoted by $\Delta_{\partial B}$, and its relationship to the Sobolev spaces $H^s(\partial B)$. We denote by $\nabla_{\partial B}$, $\text{div}_{\partial B}$, $\text{curl}_{\partial B}$, and $\overrightarrow{\text{curl}}_{\partial B}$ the surface gradient, surface divergence, scalar surface curl, and vector surface curl, respectively, and we refer to [18] or [19] for definitions of these operators. In terms of these surface differential operators the scalar Laplace-Beltrami operator is given by

$$\Delta_{\partial B} = -\text{div}_{\partial B} \nabla_{\partial B} = \text{curl}_{\partial B} \overrightarrow{\text{curl}}_{\partial B}. \tag{3.1}$$

In this definition we have introduced a negative sign (as in [16]) in order to ensure nonnegativity of the operator, and we summarize the spectral properties of $\Delta_{\partial B}$ in the following theorem (cf. [21]).

THEOREM 3.1. *There exists an orthonormal basis $\{Y_m\}_{m=0}^{\infty}$ of $L^2(\partial B)$ and a nondecreasing divergent sequence of nonnegative real numbers $\{\mu_m\}_{m=0}^{\infty}$ such that*

$$\Delta_{\partial B} Y_m = \mu_m Y_m,\ m \geq 0.$$

The first eigenpair is $\mu_0 = 0$ with $Y_0 = |\partial B|^{-1/2}$, and $\mu_m > 0$ for $m \geq 1$.

As a result of Theorem 3.1, any $\xi \in L^2(\partial B)$ may be expanded in terms of the basis $\{Y_m\}$ as

$$\xi = \sum_{m=0}^{\infty} \xi_m Y_m,\ \xi_m := \langle \xi, Y_m \rangle_{\partial B},$$

where $\langle \cdot, \cdot \rangle_{\partial B}$ denotes the inner product on $L^2(\partial B)$. From [19] it follows that, for any $s \geq 0$, the Sobolev space $H^s(\partial B)$ may be defined as

$$H^s(\partial B) := \left\{ \xi \in L^2(\partial B) \;\middle|\; \sum_{m=0}^{\infty} (1+\mu_m)^s |\xi_m|^2 < \infty \right\}, \tag{3.2}$$

equipped with the norm $\|\cdot\|_{H^s(\partial B)}$ given by

$$\|\xi\|_{H^s(\partial B)}^2 := \sum_{m=0}^{\infty} (1+\mu_m)^s |\xi_m|^2. \tag{3.3}$$

In the case $s = 0$, in which $H^0(\partial B) = L^2(\partial B)$, we denote by $\|\cdot\|_{\partial B}$ the norm on $L^2(\partial B)$ for convenience. We will use this spectral characterization of $H^s(\partial B)$ throughout our analysis.

We now proceed to define the main operator of interest. For a nonnegative number δ we define the *Bessel potential operator* $S_\delta := (I + \Delta_{\partial B})^{-\delta} : L^2(\partial B) \to L^2(\partial B)$, which may be written as

$$S_\delta \xi = \sum_{m=0}^{\infty} (1+\mu_m)^{-\delta} \xi_m Y_m, \tag{3.4}$$

where ξ has the representation (3.3). We note that this operator is well-defined since -1 is not an eigenvalue of $\Delta_{\partial B}$. A similar operator was previously used in [2, Theorem 5.1] in order to study a different type of spectral characterization of $H^s(\partial B)$. We now summarize some basic properties of the operator S_δ.

PROPOSITION 3.2. *For any $s \geq 0$, the operator S_δ is an isometric isomorphism from $H^s(\partial B)$ onto $H^{s+2\delta}(\partial B)$. In particular, $S_\delta : L^2(\partial B) \to L^2(\partial B)$ is compact whenever $\delta > 0$. Furthermore, S_δ is a positive self-adjoint operator with respect to the inner product on $L^2(\partial B)$.*

We see from the definition that S_0 coincides with the identity operator on $L^2(\partial B)$, and it follows that S_δ cannot be compact when $\delta = 0$. For later use we provide the following result concerning the summability of the sequence $\{(1+\mu_m)^{-\beta}\}$ for a given $\beta > 0$, which follows as a straightforward consequence of Weyl's law (cf. [16]).

PROPOSITION 3.3. *The sequence $\{(1+\mu_m)^{-\beta}\}$ is summable if and only if $\beta > \frac{d-1}{2}$.*

With some basic results in hand, we now proceed to the main problem of interest. We assume that $D \subseteq B$, and we introduce the auxiliary problem of finding $u_\lambda^s \in H^1_{\text{loc}}(\mathbb{R}^d \setminus \overline{B})$ satisfying

$$\Delta u_\lambda^s + k^2 u_\lambda^s = 0 \text{ in } \mathbb{R}^d \setminus \overline{B}, \tag{3.5a}$$

$$\frac{\partial u_\lambda^s}{\partial \nu} + \lambda S_\delta u_\lambda^s = -\frac{\partial u^i}{\partial \nu} - \lambda S_\delta u^i \text{ on } \partial B, \tag{3.5b}$$

$$\lim_{r\to\infty} r^{(d-1)/2} \left(\frac{\partial u_\lambda^s}{\partial r} - iku_\lambda^s \right) = 0. \tag{3.5c}$$

We remark that we have written $S_\delta u_\lambda^s$ rather than $S_\delta(u_\lambda^s|_{\partial B})$ for convenience, and for a function with a well-defined trace on ∂B (e.g. a function in $H^1(B)$) we always assume that S_δ acts in this manner. If we take $\delta = 0$, then S_0 is the identity operator and (3.5a)–(3.5c) reduces to the standard problem (2.2a)–(2.2c). In the case $\delta > 0$, the fact that S_δ is a compact and positive self-adjoint operator by Proposition 3.2 implies that (3.5a)–(3.5c) is well-posed whenever $\text{Im}(\lambda) \geq 0$.

If we choose (3.5a)–(3.5c) as the auxiliary problem in the definition of the modified far field operator $\mathcal{F}_\lambda$ and relabel it as $\mathcal{F}_\lambda^{(\delta)}$, then by the same reasoning as in the standard case (cf. [8]) we have the following result.

THEOREM 3.4. *The modified far field operator $\mathcal{F}_\lambda^{(\delta)}$ is injective with dense range provided there exists no nontrivial solution $w \in H^1(B)$ of the* δ-Stekloff problem

$$\Delta w + k^2 n w = 0 \textit{ in } B, \tag{3.6a}$$

$$\frac{\partial w}{\partial \nu} + \lambda S_\delta w = 0 \textit{ on } \partial B. \tag{3.6b}$$

We call a value of λ for which (3.6a)–(3.6b) admits a nontrivial solution a *δ-Stekloff eigenvalue*, and it will serve as our target signature for the detection of changes in n. We conclude this section with the following assumption on the wave number k, which ensures injectivity of a certain solution operator that we will introduce in Section 4. We note that this assumption is automatically satisfied whenever $\text{Im}(n) > 0$ on an open subset of B.

Assumption 3.5. We assume that k is chosen such that there exist no nontrivial solutions $\psi \in H^1(B)$ of the homogeneous Dirichlet problem

$$\Delta \psi + k^2 n \psi = 0 \text{ in B}, \tag{3.7a}$$

$$\psi = 0 \text{ on } \partial B. \tag{3.7b}$$

4. Properties of the δ-Stekloff eigenvalue problem. In this section we investigate the properties of the δ-Stekloff eigenvalues. Although many of the properties mirror those of the standard problem, we will see that some significant differences arise. We begin by introducing a nonhomogeneous version of (3.6a)–(3.6b) in which, for a given $f \in L^2(B)$ and $h \in H^{-1/2}(\partial B)$, we seek $w \in H^1(B)$ satisfying

$$\Delta w + k^2 n w = f \text{ in } B, \tag{4.1a}$$

$$\frac{\partial w}{\partial \nu} + \lambda S_\delta w = h \text{ on } \partial B. \tag{4.1b}$$

A natural question we now address is when (4.1a)–(4.1b) is well-posed. If we multiply (4.1a) by a test function, integrate by parts, and enforce the boundary condition (4.1b), then we arrive at an equivalent variational formulation of finding $w \in H^1(B)$ satisfying

$$a_\lambda^{(\delta)}(w, w') = L(w') \quad \forall w' \in H^1(B), \tag{4.2}$$

where the bounded sesquilinear form $a_\lambda^{(\delta)}(\cdot,\cdot)$ is given by

$$a_\lambda^{(\delta)}(w, w') := (\nabla w, \nabla w')_B - k^2(nw, w')_B + \lambda \langle S_\delta w, w' \rangle_{\partial B} \quad \forall w, w' \in H^1(B), \tag{4.3}$$

and the bounded antilinear functional L is given by

$$L(w') := -(f, w')_B + \langle h, w' \rangle_{\partial B} \quad \forall w' \in H^1(B). \tag{4.4}$$

Here we have used $(\cdot,\cdot)_B$ to denote the inner product on $L^2(B)$ and $\langle \cdot,\cdot \rangle_{\partial B}$ to denote the duality pairing between $H^{-1/2}(\partial B)$ and $H^{1/2}(\partial B)$ (with conjugation in the second argument). We already introduced the latter notation for the inner product on $L^2(\partial B)$, but the context should prevent any confusion in the course of our analysis.

We now reformulate (4.2) as an operator equation in $H^1(B)$, and we begin by defining the operators $\hat{\mathbb{A}}, \mathbb{B}_\lambda^{(\delta)} : H^1(B) \to H^1(B)$ by means of the Riesz representation theorem such that

$$\begin{aligned} (\hat{\mathbb{A}} w, w')_{H^1(B)} &= (\nabla w, \nabla w')_B + k^2(w, w')_B \\ (\mathbb{B}_\lambda^{(\delta)} w, w')_{H^1(B)} &= -k^2((n+1)w, w')_B + \lambda \langle S_\delta w, w' \rangle_{\partial B} \end{aligned}$$

for all $w, w' \in H^1(B)$. We note that, by these definitions, we have

$$a_\lambda^{(\delta)}(w, w') = \left(\left(\hat{\mathbb{A}} + \mathbb{B}_\lambda^{(\delta)} \right) w, w' \right)_{H^1(B)} \quad \forall w, w' \in H^1(B).$$

If we also let $\ell \in H^1(B)$ be the unique element such that $L(w') = (\ell, w')_{H^1(B)}$ for all $w' \in H^1(B)$, again by means of the Riesz representation theorem, then (4.2) may be reformulated as the operator equation

$$w \in H^1(B), \tag{4.5a}$$

$$\left(\hat{\mathbb{A}} + \mathbb{B}_\lambda^{(\delta)} \right) w = \ell. \tag{4.5b}$$

By this equivalence, we may investigate the nonhomogeneous problem (4.1a)–(4.1b) by studying the operator $\hat{\mathbb{A}} + \mathbb{B}_\lambda^{(\delta)}$. We first observe that $\hat{\mathbb{A}}$ is coercive on $H^1(B)$ and hence invertible as a result of the Riesz representation theorem. Moreover, the operator $\mathbb{B}_\lambda^{(\delta)}$ is compact due to boundedness of S_δ and the compact embeddings of $H^1(B)$ into $L^2(B)$ and $H^{1/2}(\partial B)$ into $L^2(\partial B)$, and we arrive at the following theorem and corollary.

THEOREM 4.1. *The operator $\hat{\mathbb{A}} + \mathbb{B}_\lambda^{(\delta)} : H^1(B) \to H^1(B)$ is a Fredholm operator of index zero.*

COROLLARY 4.2. *The nonhomogeneous δ-Stekloff problem (4.1a)–(4.1b) is of Fredholm type, and in particular it is well-posed whenever λ is not a δ-Stekloff eigenvalue. In this case the unique solution $w \in H^1(B)$ satisfies the estimate*

$$\|w\|_{H^1(B)} \le C \left(\|f\|_{L^2(B)} + \|h\|_{H^{-1/2}(\partial B)} \right). \tag{4.6}$$

This corollary follows immediately from Theorem 4.1 and the equivalence between (4.1a)–(4.1b) and (4.5a)–(4.5b). We now obtain our first results on the δ-Stekloff eigenvalues in the following theorem.

THEOREM 4.3. *The set of δ-Stekloff eigenvalues is discrete without finite accumulation point, and each δ-Stekloff eigenvalue has nonnegative imaginary part.*

Proof. We begin with the second assertion. If (λ, w) is an eigenpair of (3.6a)–(3.6b), then we see from taking the imaginary part of both sides of (4.2) with $L = 0$ and $w' = w$ that

$$-k^2(\mathrm{Im}(n)w, w)_B + \mathrm{Im}(\lambda)\, \langle S_\delta w, w\rangle_{\partial B} = 0.$$

We note that the second term arises from the fact that S_δ is a positive operator. It follows that

$$\mathrm{Im}(\lambda) = \frac{k^2(\mathrm{Im}(n)w, w)_B}{\langle S_\delta w, w\rangle_{\partial B}} \geq 0,$$

which we may write since S_δ is injective and $w|_{\partial B} \neq 0$ by virtue of Assumption 3.5. In addition to verifying the second assertion, this result also informs us that there are no δ-Stekloff eigenvalues in the lower half-plane, which by Theorem 4.1 implies that $\hat{\mathbb{A}} + \mathbb{B}_\lambda^{(\delta)}$ is invertible whenever $\mathrm{Im}(\lambda) < 0$. As a result of the equivalence between invertibility of $\hat{\mathbb{A}} + \mathbb{B}_\lambda^{(\delta)}$ and well-posedness of (4.1a)–(4.1b), the first assertion follows from the analytic Fredholm theorem [14, Theorem 8.26]. $\square$

We remark that many of the results established for standard Stekloff eigenvalues in [3] also hold for δ-Stekloff eigenvalues due to the favorable properties of the operator S_δ that we stated in Proposition 3.2. We now proceed to study the existence of eigenvalues, and we begin by defining the Robin-to-Dirichlet operator $T_z^{(\delta)} : L^2(\partial B) \to L^2(\partial B)$ by $T_z^{(\delta)} h := w_h^{(\delta)}|_{\partial B}$, where $w_h^{(\delta)} \in H^1(B)$ is the unique solution of (4.1a)–(4.1b) with $f = 0$ and $\lambda = z$, i.e. $w_h^{(\delta)}$ satisfies

$$\Delta w_h^{(\delta)} + k^2 n w_h^{(\delta)} = 0 \text{ in } B, \tag{4.7a}$$

$$\frac{\partial w_h^{(\delta)}}{\partial \nu} + z S_\delta w_h^{(\delta)} = h \text{ on } \partial B. \tag{4.7b}$$

Here we choose $z \in \mathbb{R}$ such that it does not coincide with any δ-Stekloff eigenvalues, which is possible since this set is discrete, and by Corollary 4.2 we see that $T_z^{(\delta)}$ is well-defined. As a consequence of (4.6) and the trace theorem we have

$$\left\| T_z^{(\delta)} h \right\|_{H^{1/2}(\partial B)} \leq C \left\| h \right\|_{L^2(\partial B)} \quad \forall h \in L^2(\partial B),$$

where C is a constant independent of h, and it follows that $T_z^{(\delta)}$ is a bounded operator from $L^2(\partial B)$ into $H^{1/2}(\partial B)$. We state the following additional boundedness result for later use, which is a direct consequence of elliptic regularity (cf. [19]).

LEMMA 4.4. *For $\delta \geq 0$ the operator $T_z^{(\delta)}$ is bounded from $H^\delta(\partial B)$ into $H^\rho(\partial B)$, where $\rho := \min\left\{\delta + 1, \frac{3}{2}\right\}$.*

In particular, the compact embedding of $H^{1/2}(\partial B)$ into $L^2(\partial B)$ implies that $T_z^{(\delta)} : L^2(\partial B) \to L^2(\partial B)$ is compact.

The operator $T_z^{(0)}$ was used in [3] and [8] to investigate the standard Stekloff eigenvalues, as λ is a Stekloff eigenvalue if and only if $(z-\lambda)^{-1}$ is an eigenvalue of $T_z^{(0)}$. For $\delta > 0$ this relationship no longer holds between δ-Stekloff eigenvalues and $T_z^{(\delta)}$, and in this case the correct operator to consider is given by

$$\Psi_z^{(\delta)} := S_{\delta/2} T_z^{(\delta)} S_{\delta/2} : L^2(\partial B) \to L^2(\partial B). \tag{4.8}$$

We first note that if $T_z^{(\delta)} h = 0$, then $w_h^{(\delta)}|_{\partial B} = 0$ and consequently $w_h^{(\delta)} = 0$ in B by Assumption 3.5. The boundary condition (4.7b) implies that $h = 0$, and it follows that $T_z^{(\delta)}$ is injective. Since the operator $S_{\delta/2}$ is injective, we conclude that $\Psi_z^{(\delta)}$ must be injective as well. We now show that the desired relationship holds between δ-Stekloff eigenvalues and the spectrum of $\Psi_z^{(\delta)}$, and for later use we prove a slightly more general result.

Proposition 4.5. *For a given $\delta \geq 0$, let δ_1, δ_2 be nonnegative numbers such that $\delta_1 + \delta_2 = \delta$. Then a given $\lambda \in \mathbb{C}$ is a δ-Stekloff eigenvalue if and only if $(z-\lambda)^{-1}$ is an eigenvalue of the operator $S_{\delta_1} T_z^{(\delta)} S_{\delta_2}$.*

Proof. We suppose that λ is a δ-Stekloff eigenvalue with eigenfunction w. If we consider the variational formulation (4.2) with $L = 0$ and add the term $(z - \lambda) \langle S_\delta w, w' \rangle_{\partial B}$ to both sides, then we see that w satisfies

$$(\nabla w, \nabla w')_B - k^2 (nw, w')_B + z \langle S_\delta w, w' \rangle_{\partial B} = (z-\lambda) \langle S_\delta w, w' \rangle_{\partial B} \quad \forall w' \in H^1(B).$$

We define $h := (z-\lambda) S_{\delta_1} w$, and we observe that

$$(\nabla w, \nabla w')_B - k^2 (nw, w')_B + z \langle S_\delta w, w' \rangle_{\partial B} = \langle S_{\delta_2} h, w' \rangle_{\partial B} \quad \forall w' \in H^1(B).$$

By definition of $T_z^{(\delta)}$ we see that $w|_{\partial B} = T_z^{(\delta)} S_{\delta_2} h$, and it follows that

$$S_{\delta_1} T_z^{(\delta)} S_{\delta_2} h = S_{\delta_1} (w|_{\partial B}) = (z-\lambda)^{-1} h.$$

Since $w|_{\partial B} \neq 0$ by Assumption 3.5 and S_{δ_1} is injective, we conclude that $(z-\lambda)^{-1}$ is an eigenvalue of $S_{\delta_1} T_z^{(\delta)} S_{\delta_2}$. As $S_{\delta_1} T_z^{(\delta)} S_{\delta_2}$ is injective, following the previous steps in reverse order yields the converse result. $\square$

As a consequence of Proposition 4.5 (with $\delta_1 = \delta_2 = \frac{\delta}{2}$) and compactness of $\Psi_z^{(\delta)}$, the spectral theorem for compact operators provides another proof that the δ-Stekloff eigenvalues are discrete without finite accumulation point, which we already established in Theorem 4.3. Since S_δ is self-adjoint, it may be easily shown (in a manner similar to [8]) that $T_z^{(\delta)}$ is self-adjoint whenever n is real-valued, from which it follows that $\Psi_z^{(\delta)}$ is self-adjoint in this case. We summarize the consequences of this observation in the following theorem.

Theorem 4.6. *If n is real-valued, then $\Psi_z^{(\delta)}$ is self-adjoint, and consequently all of the δ-Stekloff eigenvalues are real and infinitely many exist. In addition, the eigenfunctions form an orthonormal basis of $L^2(\partial B)$.*

With the exception of some technical points related to the operator S_δ, our analysis of the δ-Stekloff eigenvalue has followed the results for the standard Stekloff eigenvalue problem corresponding to $\delta = 0$. However, we previously mentioned the limited existence results for standard Stekloff eigenvalues when n is in general complex-valued (requiring that $n \in C^\infty(\overline{B})$), and we now proceed to overcome this difficulty for δ-Stekloff eigenvalues without assuming extra regularity on n. We will take advantage of a powerful result known as Lidski's theorem, and before we state the result we define the notion of a trace class operator (cf. [14] and [20]).

DEFINITION 4.7. *An operator T is a* trace class operator *on a Hilbert space if there exists a sequence of operators $\{T_m\}$ for which T_m has rank no greater than m and*

$$\sum_{m=1}^{\infty} \|T - T_m\| < \infty. \tag{4.9}$$

THEOREM 4.8. (Lidski's Theorem) *If T is a trace class operator on a Hilbert space X such that T has finite-dimensional nullspace and* $\operatorname{Im}(Tg, g)_X \geq 0$ *for every $g \in X$, then T has an infinite number of eigenvalues.*

We first remark that this theorem is not new to the field of inverse scattering theory; it has been previously used to establish that the far field operator possesses infinitely many eigenvalues when n is in general complex-valued (cf. [14]). In the present analysis we will apply this result to the operator $\Psi_z^{(\delta)}$ for sufficiently large δ depending on the dimension. Injectivity of this operator (under Assumption 3.5) implies that the nullspace of this operator has finite dimension, and in the subsequent two lemmas we verify the remaining hypotheses of Lidski's theorem.

LEMMA 4.9. *The operator $\Psi_z^{(\delta)}$ satisfies* $\operatorname{Im}\left\langle \Psi_z^{(\delta)} h, h \right\rangle_{\partial B} \geq 0$ *for all $h \in L^2(\partial B)$.*

Proof. We note that since

$$\left\langle \Psi_z^{(\delta)} h, h \right\rangle_{\partial B} = \left\langle S_{\delta/2} T_z^{(\delta)} S_{\delta/2} h, h \right\rangle_{\partial B} = \left\langle T_z^{(\delta)} S_{\delta/2} h, S_{\delta/2} h \right\rangle_{\partial B}$$

and $S_{\delta/2} h \in L^2(\partial B)$ for all $h \in L^2(\partial B)$, it suffices to show the desired result for $T_z^{(\delta)}$. Indeed, from the equivalent variational formulation of (4.7a)–(4.7b) we see that for any $h \in L^2(\partial B)$ we have

$$\begin{aligned}
\left\langle T_z^{(\delta)} h, h \right\rangle_{\partial B} &= \overline{\left\langle h, w_h^{(\delta)} \right\rangle_{\partial B}} \\
&= \overline{(\nabla w_h^{(\delta)}, \nabla w_h^{(\delta)})_B} - k^2 \overline{(n w_h^{(\delta)}, w_h^{(\delta)})_B} + z \overline{\left\langle S_\delta w_h^{(\delta)}, w_h^{(\delta)} \right\rangle_{\partial B}} \\
&= (\nabla w_h^{(\delta)}, \nabla w_h^{(\delta)})_B - k^2 (\overline{n} w_h^{(\delta)}, w_h^{(\delta)})_B + z \left\langle S_\delta w_h^{(\delta)}, w_h^{(\delta)} \right\rangle_{\partial B}.
\end{aligned}$$

We note that the third term of the final expression is unchanged by conjugation since z was chosen to be real and S_δ is a positive operator. In a similar manner we observe that the first and third terms of this expression are real, and by taking the imaginary part we conclude that

$$\operatorname{Im}\left\langle T_z^{(\delta)} h, h \right\rangle_{\partial B} = -k^2 (\operatorname{Im}(\overline{n}) w_h^{(\delta)}, w_h^{(\delta)})_B = k^2 (\operatorname{Im}(n) w_h^{(\delta)}, w_h^{(\delta)})_B \geq 0,$$

where the final inequality follows from our assumption that $\operatorname{Im}(n) \geq 0$ a.e. in D. $\square$

LEMMA 4.10. *The operator $\Psi_z^{(\delta)}$ is trace class provided that $\delta > \frac{d}{2} - 1$.*

Proof. For each $M \in \mathbb{N}_0$ we consider the operator $I^{(M)} : L^2(\partial B) \to L^2(\partial B)$ defined by

$$I^{(M)}\left(\sum_{m=0}^{\infty} \xi_m Y_m\right) := \sum_{m=0}^{M-1} \xi_m Y_m.$$

In other words, $I^{(M)}$ is the orthogonal projection of $L^2(\partial B)$ onto the span of $\{Y_0, \dots, Y_{M-1}\}$, and as a result we see that $I^{(M)}$ is a bounded linear operator on $L^2(\partial B)$ with rank M. We will show that the sequence $\{I^{(M)}\Psi_z^{(\delta)}\}$ satisfies Definition 4.7 whenever $\delta > \frac{d}{2}-1$. As a consequence of Lemma 4.4 we see that the operator $T_z^{(\delta)} S_{\delta/2}$ is bounded from $L^2(\partial B)$ into $H^\rho(\partial B)$ with $\rho = \min\left\{\delta+1, \frac{3}{2}\right\}$. For a given $h \in L^2(\partial B)$ we define $\xi_h := T_z^{(\delta)} S_{\delta/2} h$, and since $\xi_h \in H^\rho(\partial B)$ and $\{\mu_m\}$ is a nondecreasing sequence it follows that

$$\begin{aligned}
\left\|(\Psi_z^{(\delta)} - I^{(M)}\Psi_z^{(\delta)})h\right\|_{\partial B} &= \left\|S_{\delta/2}\xi_h - I^{(M)} S_{\delta/2}\xi_h\right\|_{\partial B} \\
&= \left(\sum_{m=M}^{\infty} (1+\mu_m)^{-\delta} \left|\xi_{h,m}\right|^2\right)^{1/2} \\
&= \left(\sum_{m=M}^{\infty} (1+\mu_m)^{-\delta-\rho}(1+\mu_m)^{\rho} \left|\xi_{h,m}\right|^2\right)^{1/2} \\
&\le (1+\mu_M)^{-\frac{1}{2}(\delta+\rho)} \left\|\xi_h\right\|_{H^\rho(\partial B)}.
\end{aligned}$$

By Lemma 4.4 there exists a constant C independent of h for which $\|\xi_h\|_{H^\rho(\partial B)} \le C\|h\|_{\partial B}$, and we see that

$$\left\|\Psi_z^{(\delta)} - I^{(M)}\Psi_z^{(\delta)}\right\| \le C(1+\mu_M)^{-\frac{1}{2}(\delta+\rho)} \quad \forall M \in \mathbb{N}.$$

Under the hypothesis $\delta > \frac{d}{2} - 1$ we obtain $\rho \ge \frac{d}{2}$, as ρ is defined as the minimum of two terms that are bounded below by $\frac{d}{2}$ for $d = 2, 3$. Combining these inequalities yields

$$\frac{1}{2}(\delta+\rho) > \frac{d-1}{2},$$

and from Proposition 3.3 it follows that

$$\sum_{M=0}^{\infty} (1+\mu_M)^{-\frac{1}{2}(\delta+\rho)} < \infty.$$

By the comparison theorem and Definition 4.7 we conclude that $\Psi_z^{(\delta)}$ is a trace class operator. □

We now combine the previous lemmas with Lidski's theorem in order to obtain our general existence result.

Theorem 4.11. *If $\delta > \frac{d}{2} - 1$, then there exist infinitely many δ-Stekloff eigenvalues.*

Proof. By Lemmas 4.9–4.10 we may apply Lidski's theorem in order to conclude that $\Psi_z^{(\delta)}$ has infinitely many eigenvalues whenever $\delta > \frac{d}{2} - 1$. We showed in Proposition 4.5 that there is a one-to-one correspondence between these eigenvalues and the δ-Stekloff eigenvalues, and the result follows. $\square$

4.1. Detection of δ-Stekloff eigenvalues from far field data. In this section we establish that δ-Stekloff eigenvalues can be detected from far field data, which is essential for their potential application in nondestructive testing of materials. For the standard Stekloff eigenvalues corresponding to $\delta = 0$, it has been shown in [8] and [3] that eigenvalues may be detected from far field data using either the classical linear sampling method (LSM) or the generalized linear sampling method (GLSM), respectively. It can be easily seen that our introduction of the operator S_δ does not affect any of these results, and as a consequence we only provide the relevant assumptions and theorems in order to justify the use of the generalized linear sampling method for the case $\delta > 0$. We refer to [3] for further discussion on the following assumption.

Assumption 4.12. For a given $\lambda \in \mathbb{C}$, we assume that the modified far field operator $\mathcal{F}_\lambda^{(\delta)}$ has dense range, and we assume that the problem of seeking $w \in H^1(B)$ such that

$$\Delta w + k^2 w = 0 \text{ in } B,$$
$$\frac{\partial w}{\partial \nu} + \lambda S_\delta w = 0 \text{ on } \partial B,$$

has only the trivial solution $w = 0$. The latter condition is equivalent to the assumption that λ is not a δ-Stekloff eigenvalue corresponding to $n = 1$ in D.

For a given $\lambda \in \mathbb{C}$ satisfying Assumption 4.12, we consider the functional

$$J_\alpha(\Phi_z^\infty, g) := \alpha \left| \left(F_\lambda^{(\delta)} g, g \right)_{L^2(\mathbb{S}^{d-1})} \right| + \left\| \mathcal{F}_\lambda^{(\delta)} g - \Phi_z^\infty \right\|^2_{L^2(\mathbb{S}^{d-1})},$$

where $\alpha > 0$, Φ_z^∞ is the far field pattern of the fundamental solution of the Helmholtz equation in $\mathbb{R}^d$, $g \in L^2(\mathbb{S}^{d-1})$, $F_\lambda^{(\delta)}$ is the auxiliary far field operator corresponding to the auxiliary problem (3.5a)–(3.5c), and $\mathcal{F}_\lambda^{(\delta)} = F - F_\lambda^{(\delta)}$ is the modified far field operator. The nonnegativity of J_α allows us to define the functional

$$j_\alpha(\Phi_z^\infty) := \inf_{g \in L^2(\mathbb{S}^{d-1})} J_\alpha(\Phi_z^\infty, g).$$

The following theorem serves as the justification for the generalized linear sampling method.

THEOREM 4.13. *Suppose that $\lambda \in \mathbb{C}$ satisfies Assumption 4.12, and for a given $z \in B$ consider a minimizing sequence $\{g_\alpha^{z,\lambda}\}$ defined by*

$$J_\alpha(\Phi_z^\infty, g_\alpha^{z,\lambda}) \le j_\alpha(\Phi_z^\infty) + C\alpha,$$

where $C > 0$ is a constant independent of α. Then λ is a δ-Stekloff eigenvalue if and only if the set of points z such that $\left| \left(F_\lambda^{(\delta)} g_\alpha^{z,\lambda}, g_\alpha^{z,\lambda} \right)_{L^2(\mathbb{S}^{d-1})} \right|$ is bounded as $\alpha \to 0$ is nowhere dense in B.

Theorem 4.13 suggests that we may seek eigenvalues as peaks in the graph of the indicator function

$$I(\lambda) := \limsup_{\alpha\to 0} \left| \left(F_\lambda^{(\delta)} g_\alpha^{z,\lambda}, g_\alpha^{z,\lambda} \right)_{L^2(\mathbb{S}^{d-1})} \right|,$$

possibly averaged over multiple choices of $z \in B$. In practice we consider a slightly different functional J_α that accounts for the noise in the measured far field operator F, and we regularize this functional in order to compute a minimizer rather than a minimizing sequence. We refer to [3] for further details and to [10] for an implementation of the generalized linear sampling to a different class of eigenvalue problems. We will discuss the practical application of this theorem in Section 6.

5. Stability of δ-Stekloff eigenvalues. Now that we have addressed the fundamental issue of existence of eigenvalues in Section 4, we direct our attention to the sensitivity of δ-Stekloff eigenvalues to changes in the refractive index n and in the parameter δ. We begin with two rough perturbation estimates for changes in n and δ, respectively, in a similar manner to that derived in [8]. First, we suppose that n_0 is real-valued and that (λ_0, w_0) is an eigenpair for the δ-Stelkoff problem with $n = n_0$. If n_0 is perturbed to some real-valued refractive index n_1 with $n_1 - n_0$ small, then as in [8] we see that the perturbed eigenvalue λ_1 must satisfy the rough estimate

$$\lambda_1 - \lambda_0 \approx \frac{k^2((n_1 - n_0)w_0, w_0)_B}{\langle S_\delta w_0, w_0\rangle_{\partial B}}. \tag{5.1}$$

In the case $\delta = 0$ we recover the same estimate as in [8].

By the same technique, if $(\lambda^{(\delta_j)}, w^{(\delta_j)})$ is a δ-Stekloff eigenpair for $\delta = \delta_j$, $j = 1, 2$, then it follows that

$$\lambda^{(\delta_2)} - \lambda^{(\delta_1)} \approx -\frac{\lambda^{(\delta_1)} \left\langle (S_{\delta_2} - S_{\delta_1}) w^{(\delta_1)}, w^{(\delta_1)} \right\rangle_{\partial B}}{\left\langle S_{\delta_1} w^{(\delta_1)}, w^{(\delta_1)} \right\rangle_{\partial B}}. \tag{5.2}$$

We remark that we are considering the *absolute change* in the eigenvalues rather than the relative change. The reason is that we would like to view shifts in the eigenvalues due to changes in n, and the magnitude of the eigenvalue plays no role in that analysis. We will use numerical examples in order to investigate more complicated media in Section 6, and we will continue to measure absolute change.

We have now obtained two rough estimates for the shift in an eigenvalue due to changes in n and δ. The formal asymptotic methods used to arrive at (5.1) and (5.2) are only valid for real coefficients, and as a consequence we have no such information in the case of complex-valued n. However, we are able to say something in the following subsections about the solution operator $\Psi_z^{(\delta)}$ under these perturbations, even for complex coefficients.

5.1. Stability with respect to n. We begin with perturbations of n, and we write $\Psi_{z,n}^{(\delta)}$ in order to emphasize the refractive index n. We first factorize this operator in order to make the perturbation explicit, as was done for a closely related generalized Robin eigenvalue problem in [11], and then we apply the Sobolev embedding theorem (cf. [1]) to obtain estimates on the norm of the operator $\Psi_{z,n_1}^{(\delta)} - \Psi_{z,n_0}^{(\delta)}$. We again let n_1 be the refractive index obtained by perturbing the fixed reference medium given by n_0. We define the following three operators.

(i) We define $W^{(\delta)}_{z,n_0} : L^2(\partial B) \to L^2(B)$ by $W^{(\delta)}_{z,n_0} h := w$, where $w \in H^1(B)$ satisfies (4.7a)–(4.7b) with $n = n_0$.

(ii) We define $M_{n_1,n_0} : L^2(B) \to L^2(B)$ by $M_{n_1,n_0} f := k^2(n_1 - n_0) f$.

(iii) We define $V^{(\delta)}_{z,n_1} : L^2(B) \to L^2(\partial B)$ by $V^{(\delta)}_{z,n_1} f := v|_{\partial B}$, where $v \in H^1(B)$ satisfies (4.1a)–(4.1b) with $n = n_1$ and $h = 0$.

If we let $w_1, w_0 \in H^1(B)$ satisfy (4.7a)–(4.7b) with $h = S_{\delta/2} g$ for a given $g \in L^2(\partial B)$ and $n = n_1, n_0$, respectively, then we see that $w := w_1 - w_0 \in H^1(B)$ satisfies

$$\begin{aligned} &\Delta w + k^2 n_2 w = k^2(n_2 - n_1) w_0 \text{ in } B, \\ &\frac{\partial w}{\partial \nu} + z S_\delta w = 0 \text{ on } \partial B. \end{aligned}$$

It follows from our definitions that

$$\begin{aligned} \left(\Psi^{(\delta)}_{z,n_1} - \Psi^{(\delta)}_{z,n_0}\right) g &= S_{\delta/2}(w_1|_{\partial B}) - S_{\delta/2}(w_0|_{\partial B}) \\ &= S_{\delta/2}(w|_{\partial B}) \\ &= S_{\delta/2} V^{(\delta)}_{z,n_1}\left[k^2(n_2 - n_1) w_0\right] \\ &= S_{\delta/2} V^{(\delta)}_{z,n_1} M_{n_1,n_0} W^{(\delta)}_{z,n_0} S_{\delta/2} g, \end{aligned}$$

and we obtain the factorization

$$\Psi^{(\delta)}_{z,n_1} - \Psi^{(\delta)}_{z,n_0} = S_{\delta/2} V^{(\delta)}_{z,n_1} M_{n_1,n_0} W^{(\delta)}_{z,n_0} S_{\delta/2}. \tag{5.3}$$

Our present aim is to use this factorization in order to obtain a norm estimate of $\Psi^{(\delta)}_{z,n_1} - \Psi^{(\delta)}_{z,n_0}$ in terms of the perturbation $n_1 - n_0$, and the proliferation of operators involved requires clear notation about what operator norm is being used in a given expression. As a result we denote the operator norm of a linear operator A between Banach spaces X and Y as $\|A\|_{X\to Y}$.

We first investigate the operator norm of $M_{n_1,n_0} W^{(\delta)}_{z,n_0} S_{\delta/2}$, which may be expressed as

$$\left\| M_{n_1,n_0} W^{(\delta)}_{z,n_0} S_{\delta/2} \right\|_{L^2(\partial B)\to L^2(B)} = \sup_{\substack{g\in L^2(\partial B)\\ g\neq 0}} \frac{\left\| M_{n_1,n_0} W^{(\delta)}_{z,n_0} S_{\delta/2} g \right\|_{L^2(B)}}{\|g\|_{L^2(\partial B)}}. \tag{5.4}$$

Since $S_{\delta/2}$ is an isometric isomorphism from $L^2(\partial B)$ onto $H^\delta(\partial B)$ we may instead write (5.4) in terms of $h = S_{\delta/2} g$ as

$$\begin{aligned} \left\| M_{n_1,n_0} W^{(\delta)}_{z,n_0} S_{\delta/2} \right\|_{L^2(\partial B)\to L^2(B)} &= \sup_{\substack{h\in H^\delta(\partial B)\\ h\neq 0}} \frac{\left\| M_{n_1,n_0} W^{(\delta)}_{z,n_0} h \right\|_{L^2(B)}}{\|h\|_{H^\delta(\partial B)}} \\ &= k^2 \sup_{\substack{h\in H^\delta(\partial B)\\ h\neq 0}} \frac{\left\| (n_1 - n_0) w^{(\delta)}_h \right\|_{L^2(B)}}{\|h\|_{H^\delta(\partial B)}}, \end{aligned} \tag{5.5}$$

where we have used $w^{(\delta)}_h$ to denote the solution of (4.7a)–(4.7b) with $n = n_0$ for convenience. We now wish to estimate this quantity. By the regularity results in

[6] we know that $w_h^{(\delta)} \in H^{\frac{3}{2}+\delta}(B)$ whenever $h \in H^{\delta}(\partial B)$ (for $0 \leq \delta \leq \frac{1}{2}$), and the Sobolev embedding theorem implies that $w_h^{(\delta)} \in C_b^0(B)$ (the space of bounded continuous functions in B) with continuous embedding if $\delta > 0$. In the case $\delta = 0$ and $d = 2$ we would need to apply L^p-embeddings in a similar manner, and we would obtain a different norm of $n_1 - n_0$ in the subsequent results. We see that

$$\begin{aligned}\left\|(n_1 - n_0)w_h^{(\delta)}\right\|_{L^2(B)} &= \left(\int_B |n_1 - n_0|^2 \left|w_h^{(\delta)}\right|^2 dx\right)^{1/2} \\ &\leq \left(\int_B |n_1 - n_0|^2 \, dx\right)^{1/2} \left\|w_h^{(\delta)}\right\|_{C_b^0(B)} \\ &\leq C \left\|n_1 - n_0\right\|_{L^2(B)} \|h\|_{H^\delta(\partial B)},\end{aligned}$$

where the constant C results from the embedding $H^{\frac{3}{2}+\delta}(B) \hookrightarrow C_b^0(B)$, the regularity result, and the well-posedness estimate (4.6) for $n = n_0$. In particular, this constant is independent of n_1. With this estimate in hand we see that (5.5) becomes

$$\left\|M_{n_1,n_0} W_{z,n_0}^{(\delta)} S_{\delta/2}\right\|_{L^2(\partial B)\to L^2(B)} \leq C \left\|n_1 - n_0\right\|_{L^2(B)}, \tag{5.6}$$

and from the factorization (5.3) we obtain

$$\left\|\Psi_{z,n_1}^{(\delta)} - \Psi_{z,n_0}^{(\delta)}\right\|_{L^2(\partial B)\to L^2(\partial B)} \leq C \left\|S_{\delta/2} V_{z,n_1}^{(\delta)}\right\|_{L^2(B)\to L^2(\partial B)} \left\|n_1 - n_0\right\|_{L^2(B)}. \tag{5.7}$$

The term $\|n_1 - n_0\|_{L^2(B)}$ represents the magnitude of the perturbation of n_0 (at least in the L^2-norm), but the operator norm of $S_{\delta/2} V_{z,n_1}^{(\delta)}$ still depends on n_1. We now show that this term is uniformly bounded in n_1 whenever the perturbation is small.

We begin by defining the operator $\tilde{V}_{z,n_1}^{(\delta)} : L^2(B) \to H^1(B)$ by $\tilde{V}_{z,n_1}^{(\delta)} f := v$, where $v \in H^1(B)$ satisfies (4.1a)–(4.1b) with $n = n_1$ and $h = 0$, and we note that

$$V_{z,n_1}^{(\delta)} = \Gamma_{\partial B} \tilde{V}_{z,n_1}^{(\delta)},$$

where $\Gamma_{\partial B} : H^1(B) \to L^2(\partial B)$ is the trace operator on ∂B. If $v = \tilde{V}_{z,n_1}^{(\delta)} f$ for some $f \in L^2(B)$, then it follows that v must satisfy

$$\begin{aligned}&\Delta v + k^2 n_0 v = f - k^2(n_1 - n_0)v \text{ in } B, \\ &\frac{\partial v}{\partial \nu} + z S_\delta v = 0 \text{ on } \partial B,\end{aligned}$$

which implies that

$$\tilde{V}_{z,n_1}^{(\delta)} f = v = \tilde{V}_{z,n_0}^{(\delta)} \left[f - k^2(n_1 - n_0)v\right] = \left(\tilde{V}_{z,n_0}^{(\delta)} - \tilde{V}_{z,n_0}^{(\delta)} M_{n_1,n_0} \tilde{V}_{z,n_1}^{(\delta)}\right) f. \tag{5.8}$$

Since (5.8) holds for all f we see that

$$\left(I + \tilde{V}_{z,n_0}^{(\delta)} M_{n_1,n_0}\right) \tilde{V}_{z,n_1}^{(\delta)} = \tilde{V}_{z,n_0}^{(\delta)}. \tag{5.9}$$

Elliptic regularity implies that range$(\tilde{V}_{z,n_1}^{(\delta)}) \subseteq H^2(B)$, and consequently we may consider $I + \tilde{V}_{z,n_0}^{(\delta)} M_{n_1,n_0}$ as an operator from $H^2(B)$ to itself. It follows that if

$$\|M_{n_1,n_0}\|_{H^2(B)\to L^2(B)} < \left\|\tilde{V}_{z,n_0}^{(\delta)}\right\|_{L^2(B)\to H^2(B)}^{-1} \tag{5.10}$$

then the operator $I + \tilde{V}_{z,n_0}^{(\delta)} M_{n_1,n_0}$ is invertible, and from a Neumann series expansion we obtain the estimate

$$\left\| \tilde{V}_{z,n_1}^{(\delta)} \right\|_{L^2(B)\to H^2(B)} \leq \frac{\left\| \tilde{V}_{z,n_0}^{(\delta)} \right\|_{L^2(B)\to H^2(B)}}{1 - \left\| \tilde{V}_{z,n_0}^{(\delta)} \right\|_{L^2(B)\to H^2(B)} \left\| M_{n_1,n_0} \right\|_{H^2(B)\to L^2(B)}}. \tag{5.11}$$

Since $H^2(B)$ is continuously embedded into $C_b^0(B)$ by the Sobolev embedding theorem, we may take a similar approach as we did above in order to show that

$$\left\| M_{n_1,n_0} \right\|_{H^2(B)\to L^2(B)} \leq C \left\| n_1 - n_0 \right\|_{L^2(B)}, \tag{5.12}$$

where the constant C depends only upon the continuous embedding of $H^2(B)$ into $C_b^0(B)$. In particular, this constant is independent of n_1. Thus, we observe that for sufficiently small $\|n_1 - n_0\|_{L^2(B)}$ the condition (5.10) is satisfied and (5.11) follows in turn. The result is that $\left\| \tilde{V}_{z,n_1}^{(\delta)} \right\|_{L^2(B)\to H^2(B)}$ is uniformly bounded in this case, and since $S_{\delta/2}$ and $\Gamma_{\partial B}$ are bounded and independent of n_1 we obtain the following theorem as a direct consequence of (5.7).

THEOREM 5.1. *Assume that $\delta > 0$. If n_0 is fixed and $\|n_1 - n_0\|_{L^2(B)}$ is sufficiently small, then there exists a constant C_{n_0} independent of n_1 such that*

$$\left\| \Psi_{z,n_1}^{(\delta)} - \Psi_{z,n_0}^{(\delta)} \right\|_{L^2(\partial B)\to L^2(\partial B)} \leq C_{n_0} \left\| n_1 - n_0 \right\|_{L^2(B)}. \tag{5.13}$$

5.2. Stability with respect to δ. We now consider stability with respect to the parameter δ. Unlike perturbations of n, we have complete control over the value of δ, and as a result the main role of this analysis is to compare the cases $\delta > 0$ and $\delta = 0$. For the remainder of this section we will primarily work with Sobolev spaces on the boundary ∂B, and for ease of notation we define $\|A\|_{s,t} := \|A\|_{H^s(\partial B)\to H^t(\partial B)}$ for a bounded linear operator $A : H^s(\partial B) \to H^t(\partial B)$. We recall that $H^0(\partial B) = L^2(\partial B)$ by definition.

Our goal in this section is to establish that the δ-Stekloff eigenvalues converge to the standard Stekloff eigenvalues as $\delta \to 0^+$. Since $\Psi_z^{(\delta)}$ is bounded for $\delta \geq 0$, this result would follow from showing that $\Psi_z^{(\delta)} \to \Psi_z^{(0)}$ in operator norm as $\delta \to 0^+$ (cf. [17]). However, for technical reasons we instead consider the operator $\tilde{\Psi}_z^{(\delta)} := S_\delta T_z^{(\delta)}$, whose eigenvalues coincide with those of $\Psi_z^{(\delta)}$ as a consequence of Proposition 4.5 (with $\delta_1 = \delta$ and $\delta_2 = 0$). We begin with the following technical lemma.

LEMMA 5.2. *The operator $S_\delta : H^{1/2}(\partial B) \to L^2(\partial B)$ converges in operator norm to the inclusion operator $I : H^{1/2}(\partial B) \to L^2(\partial B)$ as $\delta \to 0^+$.*

Proof. For a given $\xi \in H^{1/2}(\partial B)$ we see that

$$\begin{aligned} \|(S_\delta - I)\xi\|_{L^2(\partial B)} &= \left\| \sum_{m=0}^{\infty} \left[(1+\mu_m)^{-\delta} - 1 \right] \xi_m Y_m \right\|_{L^2(\partial B)} \\ &= \left(\sum_{m=0}^{\infty} \left[1 - (1+\mu_m)^{-\delta} \right]^2 |\xi_m|^2 \right)^{1/2} \\ &= \left(\sum_{m=0}^{\infty} \left[\frac{1 - (1+\mu_m)^{-\delta}}{(1+\mu_m)^{1/4}} \right]^2 (1+\mu_m)^{1/2} |\xi_m|^2 \right)^{1/2}. \end{aligned} \tag{5.14}$$

We have introduced the term $(1+\mu_m)^{1/2}$ into this equation in order to obtain the $H^{1/2}(\partial B)$-norm of ξ, and we now seek to control the term $\frac{1-(1+\mu_m)^{-\delta}}{(1+\mu_m)^{1/4}}$. While this sequence is not monotonic in m, we now show that it is monotonic eventually, i.e. for $m \geq m_*$, and more importantly that m_* may be chosen independently of δ. We consider the function

$$\varphi_\delta(t) := \frac{1-t^{-\delta}}{t^{1/4}}, \quad t \in [1,\infty),$$

and we observe that φ_δ is decreasing on the interval $\big((4\delta+1)^{1/\delta},\infty\big)$ for each $\delta > 0$. Further calculations show that

$$\sup_{\delta>0}(4\delta+1)^{1/\delta} = \lim_{\delta\to 0}(4\delta+1)^{1/\delta} = e^4,$$

and we conclude that φ_δ is decreasing on (e^4,∞) for all $\delta > 0$. Since the sequence $\{\mu_m\}$ increases without bound we may choose $m_* \in \mathbb{N}_0$ such that $1+\mu_{m_*} > e^4$, and it follows that $\left\{\frac{1-(1+\mu_m)^{-\delta}}{(1+\mu_m)^{1/4}}\right\}$ is a non-increasing sequence for $m \geq m_*$. As a consequence we may split the final series in (5.14) by this index in order to obtain

$$\|(S_\delta - I)\xi\|_{L^2(\partial B)} \leq \max_{0\leq m\leq m_*}\left\{\frac{1-(1+\mu_m)^{-\delta}}{(1+\mu_m)^{1/4}}\right\}\|\xi\|_{H^{1/2}(\partial B)},$$

which implies the norm bound

$$\|S_\delta - I\|_{\frac{1}{2},0} \leq \max_{0\leq m\leq m_*}\left\{\frac{1-(1+\mu_m)^{-\delta}}{(1+\mu_m)^{1/4}}\right\}. \tag{5.15}$$

Since

$$\lim_{\delta\to 0}\frac{1-(1+\mu_m)^{-\delta}}{(1+\mu_m)^{1/4}} = 0$$

for each $m \in \mathbb{N}_0$ and the maximum in (5.15) is taken over finitely many such terms, we conclude that $\|S_\delta - I\|_{\frac{1}{2},0} \to 0$ as $\delta \to 0^+$. □

THEOREM 5.3. *The operator* $\tilde{\Psi}_z^{(\delta)} : L^2(\partial B) \to L^2(\partial B)$ *converges in operator norm to* $\tilde{\Psi}_z^{(0)} : L^2(\partial B) \to L^2(\partial B)$ *as* $\delta \to 0^+$.

Proof. We first observe that by the triangle inequality we have

$$\left\|\tilde{\Psi}_z^{(\delta)} - \tilde{\Psi}_z^{(0)}\right\|_{0,0} \leq \|S_\delta\|_{\frac{1}{2},0}\left\|T_z^{(\delta)} - T_z^{(0)}\right\|_{0,\frac{1}{2}} + \|S_\delta - I\|_{\frac{1}{2},0}\left\|T_z^{(0)}\right\|_{0,\frac{1}{2}}. \tag{5.16}$$

From Lemma 5.2 we already know that $\|S_\delta - I\|_{\frac{1}{2},0} \to 0$ as $\delta \to 0^+$, which also implies that $\|S_\delta\|_{\frac{1}{2},0}$ is bounded as $\delta \to 0^+$. Thus, it suffices to show that $\left\|T_z^{(\delta)} - T_z^{(0)}\right\|_{0,\frac{1}{2}} \to 0$ as $\delta \to 0^+$, and this pursuit comprises the remainder of the proof. For a given $h \in L^2(\partial B)$ we consider the solution $w_h^{(\delta)}$ of (4.7a)–(4.7b) for each $\delta \geq 0$, and we recall that $w_h^{(\delta)}|_{\partial B} = T_z^{(\delta)}h$. For $\delta > 0$ we see that $w := w_h^{(\delta)} - w_h^{(0)} \in H^1(B)$ satisfies

$$\Delta w + k^2 n w = 0 \text{ in } B,$$

$$\frac{\partial w}{\partial \nu} + zS_\delta w = z(I - S_\delta)w_h^{(0)} \text{ on } \partial B,$$

and by definition of $T_z^{(\delta)}$ we obtain

$$(T_z^{(\delta)} - T_z^{(0)})h = w|_{\partial B} = T_z^{(\delta)}\left[z(I - S_\delta)T_z^{(0)}h\right] = zT_z^{(\delta)}(I - S_\delta)T_z^{(0)}h.$$

Since this result holds for all $h \in L^2(\partial B)$ we arrive at the factorization

$$T_z^{(\delta)} - T_z^{(0)} = zT_z^{(\delta)}(I - S_\delta)T_z^{(0)}, \tag{5.17}$$

and we immediately obtain the estimate

$$\left\|T_z^{(\delta)} - T_z^{(0)}\right\|_{0,\frac{1}{2}} \le |z| \left\|T_z^{(\delta)}\right\|_{0,\frac{1}{2}} \|I - S_\delta\|_{\frac{1}{2},0} \left\|T_z^{(0)}\right\|_{0,\frac{1}{2}}. \tag{5.18}$$

If $z = 0$, then the operator $T_z^{(\delta)}$ is clearly independent of δ and we have nothing to prove. Thus, we assume that $z \neq 0$. Since $\|I - S_\delta\|_{\frac{1}{2},0} \to 0$ as $\delta \to 0^+$ we need only show that $\left\|T_z^{(\delta)}\right\|_{0,\frac{1}{2}}$ is bounded as $\delta \to 0^+$. We may rewrite (5.17) as

$$T_z^{(\delta)}\left[I - z(I - S_\delta)T_z^{(0)}\right] = T_z^{(0)}, \tag{5.19}$$

and since $\|I - S_\delta\|_{\frac{1}{2},0} \to 0$ as $\delta \to 0^+$ we see that $\delta > 0$ may be taken sufficiently small in order to guarantee that

$$\|I - S_\delta\|_{\frac{1}{2},0} < |z|^{-1} \left\|T_z^{(0)}\right\|_{0,\frac{1}{2}}^{-1}.$$

In this case the operator $I - z(I - S_\delta)T_z^{(0)}$ is invertible, and it follows from (5.19) and a Neumann series expansion that

$$\left\|T_z^{(\delta)}\right\|_{0,\frac{1}{2}} \le \frac{\left\|T_z^{(0)}\right\|_{0,\frac{1}{2}}}{1 - |z| \left\|T_z^{(0)}\right\|_{0,\frac{1}{2}} \|I - S_\delta\|_{\frac{1}{2},0}}.$$

Thus, we see that $\left\|T_z^{(\delta)}\right\|_{0,\frac{1}{2}}$ is bounded as $\delta \to 0^+$, and from (5.18) we conclude that $\left\|T_z^{(\delta)} - T_z^{(0)}\right\|_{0,\frac{1}{2}} \to 0$ as $\delta \to 0^+$. $\square$

The importance of Theorem 5.3 lies in the observation that the δ-Stekloff eigenvalues must converge to the standard Stekloff eigenvalues as $\delta \to 0^+$, and consequently the δ-Stekloff eigenvalue problem may be viewed as a perturbation of the standard problem. However, we note that this result does not imply the existence of infinitely many standard Stekloff eigenvalues for $\mathbb{R}^2$, even though the result holds for δ-Stekloff eigenvalues whenever $\delta > 0$. As a final note we remark that from (5.15) it can be shown that $\|S_\delta - I\|_{\frac{1}{2},0} = O(\delta)$ as $\delta \to 0^+$, and in the proof of Theorem 5.3 we see that this convergence rate holds for $\left\|\tilde{\Psi}_z^{(\delta)} - \tilde{\Psi}_z^{(0)}\right\|_{0,0}$ as well.

6. Numerical examples . Up to this point we have examined the theoretical properties of δ-Stekloff eigenvalues, and in particular we established the existence of infinitely many eigenvalues and their convergence to the standard Stekloff eigenvalues. We also showed that the eigenvalues are stable with respect to changes in the coefficient n, but beyond the rough perturbation estimate (5.1) for real-valued n we have no information on the sensitivity of any particular eigenvalue. In this section we investigate this question through a series of numerical examples, but we first introduce a further modification of δ-Stekloff eigenvalues that provides a greater level of control over their sensitivity to changes in n.

For some $\sigma > 0$ we define the operator $S_{\delta,\sigma} : L^2(\partial B) \to L^2(\partial B)$ by $S_{\delta,\sigma} := (I + \Delta_{\partial B} + \sigma P_0)^{-1}$, where $P_0 : L^2(\partial B) \to L^2(\partial B)$ is the orthogonal projection onto $\ker(\Delta_{\partial B}) = \operatorname{span}\{Y_0\}$. We may write this operator explicitly as

$$S_{\delta,\sigma}\xi = (1+\sigma)^{-\delta}\xi_0 Y_0 + \sum_{m=1}^{\infty}(1+\mu_m)^{-\delta}\xi_m Y_m. \tag{6.1}$$

We note that the only change is to the lowest order Fourier coefficient of ξ in the eigenbasis $\{Y_m\}$, and consequently all of the results that we have proved for S_δ still hold, with the exception that $S_{\delta,\sigma}$ is no longer an isometry from $H^s(\partial B)$ onto $H^{s+2\delta}(\partial B)$. However, we could equivalently characterize the Sobolev spaces $H^s(\partial B)$ in terms of the spectrum of $\Delta_{\partial B} + \sigma P_0$ in order to recover this result.

The motivation for this modification of S_δ is that, for case of $B = D$ with D chosen to be the unit disk in $\mathbb{R}^2$ and constant n, the δ-Stekloff eigenvalues form a sequence $\{\lambda_m^{(\delta)}\}_{m\geq 0}$ given by

$$\lambda_m^{(\delta)} = -\frac{k\sqrt{n}J_m'(k\sqrt{n})}{(1+m^2)^{-\delta}J_m(k\sqrt{n})} = (1+m^2)^{\delta}\lambda_m^{(0)}, \tag{6.2}$$

where J_m is the Bessel function of the first kind of order m and $\{\lambda_m^{(0)}\}$ is the sequence of standard Stekloff eigenvalues (cf. [8] for the computation with $\delta = 0$). We note that this sequence includes multiplicity of the eigenvalues, which is 1 for $m = 0$ (with eigenfunction 1) and 2 for $m > 0$ (with eigenfunctions given by the nonconstant spherical harmonics $\{Y_m\}$, which are $\{e^{-im\theta}, e^{im\theta}\}_{m>0}$ for $d = 2$). A similar result in terms of the three-dimensional spherical harmonics can also be seen to hold for the case of a ball in $\mathbb{R}^3$. It follows that the shift in an eigenvalue $\lambda_m^{(\delta)}$ is amplified by a factor of $(1+m^2)^\delta$ compared to the corresponding standard Stekloff eigenvalue $\lambda_m^{(0)}$. However, for $m = 0$ we have $(1+m^2)^\delta = 1$ and the shift is identical. This observation is significant because for $\delta = 0$ the lowest order eigenvalue $\lambda_0^{(0)}$ is most easily detected from far field data and has been noted as being the eigenvalue most robust to noisy data (cf. [8]), but for $\delta > 0$ the corresponding δ-Stekloff eigenvalue $\lambda_0^{(\delta)}$ does not benefit from any amplification in its sensitivity. We will justify the detectability of the most sensitive δ-Stekloff eigenvalue for two domains in Section 6.5. By introducing the parameter $\sigma > 0$ we improve this amplication factor to $(1+\sigma)^\delta > 1$ for the unit disk, and we will provide examples in this section in which we clearly see this amplification factor for more complicated domains. With the exception of Section 6.4, we choose $\sigma = 1$ for the remainder of our discussion, and we will continue to refer to the associated eigenvalues as δ-Stekloff eigenvalues.

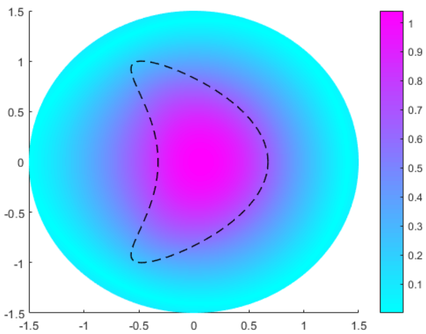
6.1. Description of numerical examples. We restrict our attention to numerical examples in $\mathbb{R}^2$, and we compute eigenvalues and eigenfunctions using the finite element method. We apply a standard finite element discretization of the δ-Stekloff eigenvalue problem (4.2) (with $L = 0$) with $\mathbb{P}_1$ finite elements using `FreeFem++` [15], read the appropriate matrices into `MATLAB`, and use the built-in `eigs` function. We discretize the operator $S_{\delta,\sigma}$ using the finite element nodal basis, and we truncate to 101 Fourier coefficients in the Laplace-Beltrami eigenbasis. In order to provide a direct comparison with the results in [8], we perform our examples with $k = 1$ and a constant value of $n = 4$ in D, for which we consider two possibilities. First, we choose the support D to be in the shape of a **kite** parametrized by

$$\mathbf{r}(t) = (0.5(\cos(t) + 0.65\cos(2t) - 0.3), \sin(t)), \quad 0 \le t \le 2\pi.$$

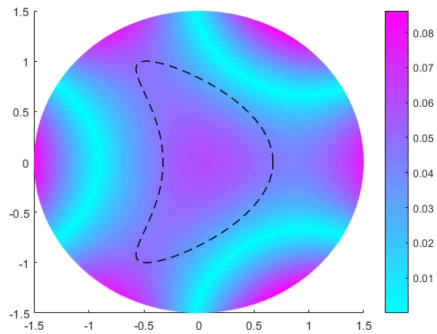
Second, we choose the support D to be an **L-shaped domain** given by removing the square $[0.1, 1.1] \times [-1.1, -0.1]$ from the square $[-0.9, 1.1] \times [-1.1, 0.9]$. We choose B to be the disk of radius 1.5 centered at the origin, which contains both the kite and L-shaped domains as required. We refer to Figure 6.1 for a graphical representation of each domain and for a plot of two eigenfunctions corresponding to each choice of D. We note that the eigenfunctions have been normalized such that $\|w\|_{L^2(\partial B)} = 1$. We will see that the eigenfunctions displayed in the left column correspond to the eigenvalues that are most sensitive to the overall change in n we will consider in the first example. As these eigenfunctions have large magnitude throughout most of D and they have small boundary values, this result agrees with what we would expect from the rough perturbation estimate (5.1).

6.2. Sensitivity of eigenvalues to overall changes in n. We first examine the sensitivity of the eigenvalues to an overall change in the constant refractive index $n = 4$. In Figure 6.2 we plot the magnitude of the absolute shift in each eigenvalue as n ranges from the reference value of $n = 4$ to $n = 4.1$. We perform this test for both the kite and L-shaped domain with $\delta = 0$ and $\delta = 1$. We observe that at least one eigenvalue displays a significant shift in all four cases, but in the case $\delta = 1$ this shift is much greater for the most sensitive eigenvalue ($\lambda_{15} = -24.0928$ for the kite domain and $\lambda_{17} = -8.52$ for the L-shaped domain). In particular, the shift is amplified by a factor of 2 in both cases, as we would expect in the case of the unit disk that we discussed at the beginning of this section. We have observed in further testing that the sensitivity may be improved by choosing higher values of δ and σ.

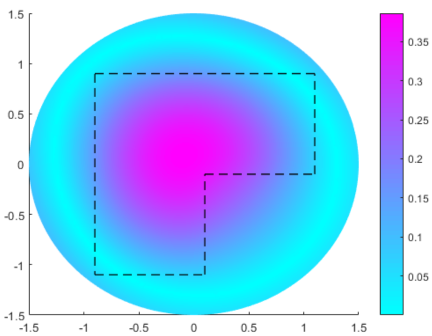
6.3. Sensitivity of eigenvalues to circular flaws in n. We now examine the sensitivity of the eigenvalues to a circular flaw in the constant refractive index $n = 4$, and for consistency we adopt the same approach taken in [8] and [12]. In particular, given a point (x_c, y_c) and a positive number r_c, we define the flawed refractive index n_c in D such that $n_c = 1$ in the disk of radius r_c centered at (x_c, y_c) and $n_c = n = 4$ otherwise. This choice is meant to simulate the detection of a void or cavity that has formed in a medium. For both the kite and the L-shaped domain we fix $r_c = 0.05$ and $y_c = 0$, and we consider $-0.2 \le x_c \le -0.6$ for the kite and $-0.4 \le x_c \le 0.8$ for the L-shaped domain. In Figure 6.3 we plot the magnitude of the absolute shift in each eigenvalue as x_c varies. We perform this test for both $\delta = 0$ and $\delta = 1$. We observe that at least one eigenvalue displays a significant shift in all four cases, but in the case $\delta = 1$ this shift is much greater for the most sensitive eigenvalue. As in the previous example, the shift is amplified by a factor of 2. We notice that in Figure 6.3d some of the curves display a great deal of variation over the domain. This observation is likely explained by the rather complicated eigenfunctions for the L-shaped domain, as seen in Figure 6.1d.



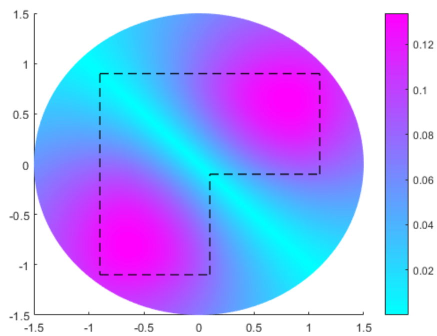
(a) kite, $\lambda = -24.0928$



(b) kite, $\lambda = -17.8471$



(c) L-shaped domain, $\lambda = -8.52$

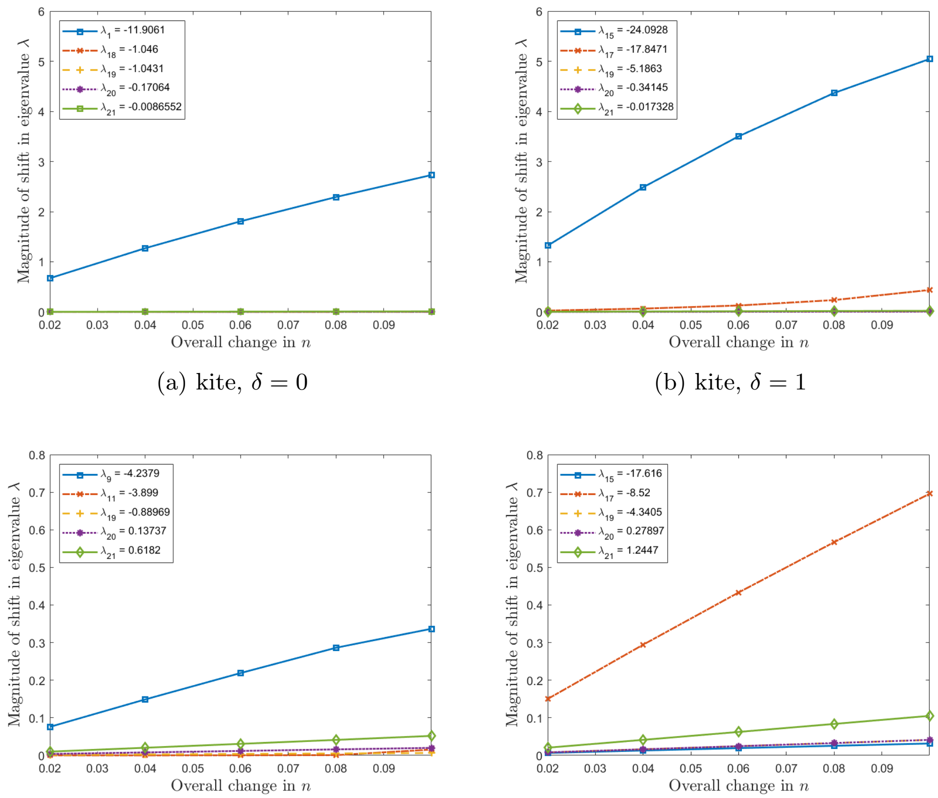


(d) L-shaped domain, $\lambda = 1.2447$

Fig. 6.1: Plots showing two δ-Stekloff eigenfunctions for the kite (top row) and the L-shaped domain (bottom row) with $\delta = 1$.

6.4. The influence of σ on the eigenvalues. In Figure 6.4 we show the δ-Stekloff eigenvalues corresponding to multiple values of σ, namely $\sigma = 0, 1, 2, 3, 4$, for $n = 4$ and $n = 4.1$. We observe that most of the eigenvalues do not exhibit a noticeable shift, as seen in Figure 6.2, and these eigenvalues also appear to be unaffected by changes in σ. However, the eigenvalues noted to be most sensitive to changes in n in Figures 6.2 and 6.3 can be seen to depend strongly on σ, as they translate to the right as σ increases. More importantly, we see that the gap between these eigenvalues for $n = 4$ and $n = 4.1$ increases as σ increases, which clarifies the role of σ as an amplification factor on the sensitivity of these eigenvalues.

6.5. Detection of eigenvalues from far field data. Finally, we justify our assertion that the most sensitive eigenvalue is detectable from far field data, and we return to the case $\sigma = 1$ for the remainder of our discussion. In Section 4.1 we provided the relevant results from [3] that justify the use of the generalized linear sampling method (GLSM) for detecting δ-Stekloff eigenvalues from far field data, and we referred to [8] for detection using the classical linear sampling method (LSM). The central idea is to construct an indicator function that may be evaluated over a range

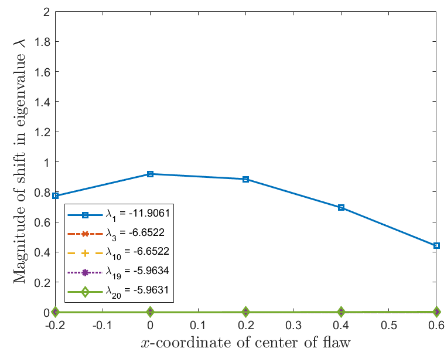


(c) L-shaped domain, $\delta = 0$ (d) L-shaped domain, $\delta = 1$

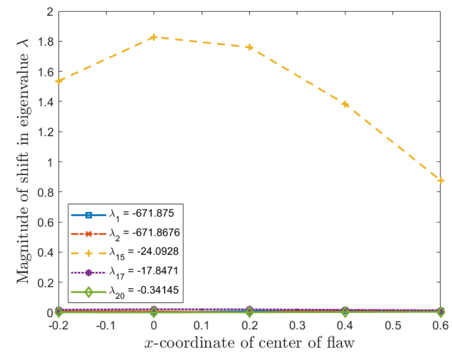
Fig. 6.2: Plots showing the magnitude of the shift in the standard Stekloff eigenvalues (left column) and δ-Stekloff eigenvalues with $\delta = 1$ (right column) for both a kite (top row) and an L-shaped domain (bottom row) due to an overall change in the refractive index n. We observe that the maximum sensitivity is significantly greater for $\delta = 1$.

of the eigenparameter λ and which gives large values near an eigenvalue and small values away from an eigenvalue. We then seek eigenvalues as peaks in the graph of this indicator function. We refer to [3] and [10] for examples of this implementation, and we denote the resulting indicator function as $I(g_\lambda^{\text{glsm}})$ for each $\lambda \in \mathbb{C}$ considered.

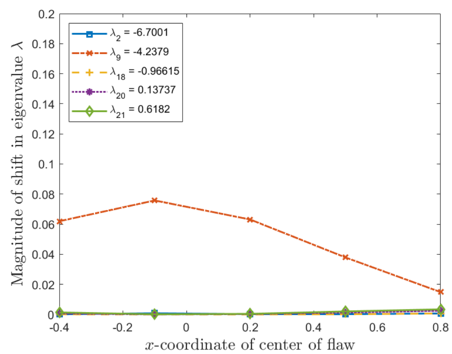
For simplicity we show in Figure 6.5 the detection of the most sensitive eigenvalue for each choice of D (based on the results of Figures 6.2 and 6.3) using the linear sampling method described in [8]. We plot the eigenvalues obtained using the finite element method with red markers for comparison, and we see that the peaks of the LSM indicator function agree with many of the eigenvalues. In particular, the largest peak corresponds to the most sensitive eigenvalue for each choice of D ($\lambda_{15} = -24.0928$ for the kite domain and $\lambda_{17} = -8.52$ for the L-shaped domain). For each example we have applied multiplicative uniform noise as in [8], which leads to a 1% relative error in the synthetic far field data. The presence of noise often leads to missed eigenvalues (cf. [8], [10], [12], and [13]) as we see in Figure 6.5a. In the absence of noise we



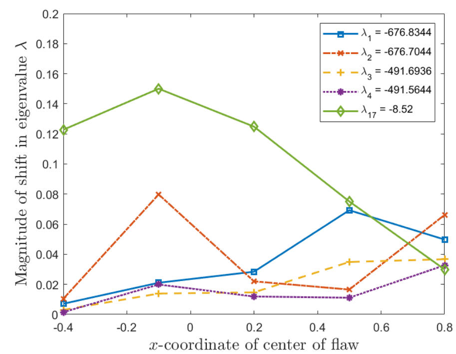
(a) kite, $\delta = 0$



(b) kite, $\delta = 1$



(c) L-shaped domain, $\delta = 0$



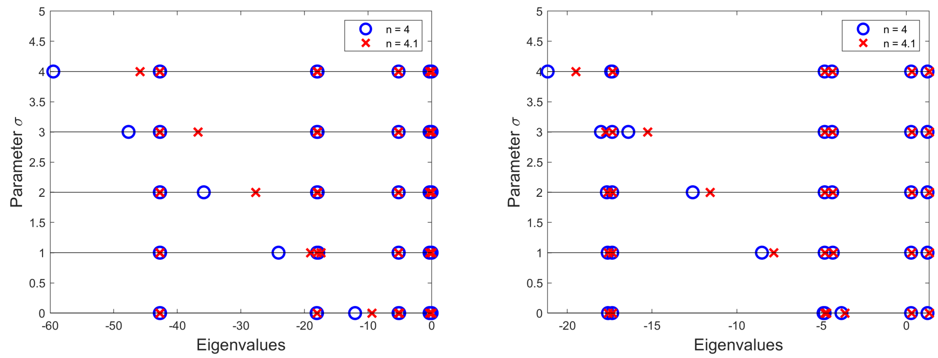
(d) L-shaped domain, $\delta = 1$

Fig. 6.3: Plots showing the magnitude of the shift in the standard Stekloff eigenvalues (left column) and δ-Stekloff eigenvalues with $\delta = 1$ (right column) for both a kite (top row) and an L-shaped domain (bottom row) due to a circular flaw in the refractive index n. We observe that the maximum sensitivity is significantly greater for $\delta = 1$.

have observed that all of the eigenvalues in this range are detected, and even higher levels of noise still allow for a reliable detection of the most sensitive eigenvalue in each case. We note that for each λ the auxiliary problem (3.5a)–(3.5c) was computed using separation of variables, which is possible in this case since B is a disk.

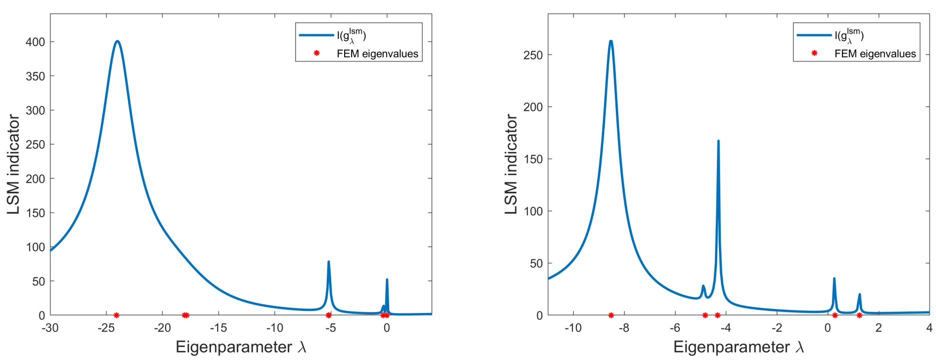
We see that for the simple examples we have considered the δ-Stekloff eigenvalues (with $\sigma = 1$) display a greater shift due to changes in n than the standard Stekloff eigenvalues corresponding to $\delta = 0$, and we have demonstrated that they can be detected from far field data. We conclude this section by remarking that for a constant complex-valued n we obtain similar results to the examples we have considered.

7. Conclusion. In this paper we have introduced a modification of the Stekloff eigenvalue problem involving an operator S_δ related to the Laplace-Beltrami operator on the boundary of the domain, and we have shown that this new δ-Stekloff problem overcomes two persistent difficulties associated with this class of eigenvalue problems. First, we applied Lidski's theorem in order to prove that infinitely many δ-Stekloff



(a) kite, $\delta = 1$ (b) L-shaped domain, $\delta = 1$

Fig. 6.4: Plots showing the δ-Stekloff eigenvalues for multiple values of σ for the kite (left) and the L-shaped domain (right) with $\delta = 1$ and both $n = 4$ (blue circles) and $n = 4.1$ (red $\times$ symbols). The y-axis represents the value of σ, and the x-axis represents the δ-Stekloff eigenvalues corresponding to each value of σ. Horizontal black lines mark the values of σ considered.



(a) kite, $\delta = 1$ (b) L-shaped domain, $\delta = 1$

Fig. 6.5: Plots showing the detection of the most sensitive eigenvalue from far field data for the kite (left) and the L-shaped domain (right) with $n = 4$ and $\delta = 1$. The eigenvalues computed by the finite element method are shown as red markers for comparison.

eigenvalues exist when δ is above a certain threshold depending on the dimension. Second, through a series of numerical examples we observed that with $\delta > 0$ and the introduction of a suitable amplification parameter σ the δ-Stekloff eigenvalues display a significantly greater response to changes in the refractive index of an inhomogeneous medium than that observed for the standard Stekloff eigenvalues.

We note that the operator $S_{\delta,\sigma}$ essentially modulates the Fourier coefficients of a certain term in the boundary condition of the new auxiliary problem. Interestingly, we achieved the first result by *reducing* the magnitudes of the Fourier coefficients in a suitably rapid manner at the tail of the sequence, i.e. for all but finitely many coefficients, and we achieved the second result by *increasing* the magnitude of the first Fourier coefficient. We have mentioned that other auxiliary scattering problems have been introduced which include a parameter γ that may be tuned to increase the sensitivity of the eigenvalues (cf. [3, 10, 12, 13]), but the effect of tuning δ and σ in the present problem is much more straightforward.

As we indicated in the introduction, this idea is not restricted to the Stekloff eigenvalue problem for an inhomogeneous medium. The most straightforward generalization is to scattering by an anisotropic medium, in which the main equation in the physical scattering problem in D is replaced by

$$\nabla \cdot A\nabla w + k^2 nw = 0 \text{ in } D,$$

where A is a $d \times d$ matrix-valued function with certain restrictions (cf. [3]). Many of the results we have established would still hold in this case, but some differences would arise due to the regularity results we used.

The entirety of our discussion has been devoted to a particular example of a tailored eigenvalue method in which we designed the eigenvalue problem to guarantee existence and sensitivity of the eigenvalues, but it is by no means the only choice. It may be possible to design auxiliary problems which provide improved results for specific materials or types of scattering, and this idea is an avenue of future research.

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