A New Modified Transmission Eigenvalue Problem for Electromagnetic Scattering

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***Abstract*—We introduce a new eigenvalue problem arising from electromagnetic scattering which may potentially be used as a target signature in nondestructive testing of materials. After establishing some basic properties of this eigenvalue problem, we show that the eigenvalues may be determined from measured scattering data.**

***Index Terms*—inverse scattering, nondestructive testing, eigenvalue problems**

I. Introduction

A recent subject of interest has been the mathematical study of eigenvalue problems arising in scattering theory and their potential use as a target signature in nondestructive testing of materials. The main objective of nondestructive testing is to detect flaws or other changes in a material without inflicting damage upon it, and in the present context we aim to do so by interrogating the medium with electromagnetic radiation and measuring the resulting scattered field. In principle, one should be able to determine the constitutive parameters of the material from this data and directly observe any flaws. However, for anisotropic materials, in which the constitutive parameters of the material depend on the direction of measurement, it is known that multiple configurations of these parameters may yield the same measured scattering data [1].

Thus, we restrict ourselves to detecting changes in the material relative to some known reference configuration. We note that methods exist to determine the support of the medium, such as the linear sampling method [2], and consequently we only seek to detect changes in the constitutive parameters of the material. Our main tool is an eigenvalue problem that we will introduce shortly, in which the eigenvalue serves as a *target signature*. In order to detect a change in a tested material, we compute the eigenvalues and compare them to the known eigenvalues for the reference material, and we infer changes from significant shifts in these values.

The layout of our discussion is as follows. We will begin in Section II by introducing the scattering problem and the eigenvalue problem of interest, after which we state some important mathematical properties of the eigenvalues. In Section III we provide the result that allows eigenvalues to be computed from measured scattering data, and we provide an example that shows the practical application of the eigenvalues that we outlined above.

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II. The Modified Transmission Eigenvalue Problem

We consider scattering of a time-harmonic incident electromagnetic field $(\mathbf{E}^i, \mathbf{H}^i)$ by an inhomogeneous medium with relative electric permittivity ϵ and relative magnetic permeability μ, which may be modeled in terms of the electric field as the problem of seeking a total field $\mathbf{E}$ and a scattered field $\mathbf{E}^s$ satisfying

$$\nabla \times \mu^{-1} \nabla \times \mathbf{E} - k^2 \epsilon \mathbf{E} = \mathbf{0} \text{ in } D, \tag{1a}$$

$$\nabla \times \nabla \times \mathbf{E}^s - k^2 \mathbf{E}^s = \mathbf{0} \text{ in } \mathbb{R}^3 \setminus \overline{D}, \tag{1b}$$

$$\boldsymbol{\nu} \times \mathbf{E} - \boldsymbol{\nu} \times \mathbf{E}^s = \boldsymbol{\nu} \times \mathbf{E}^i \text{ on } \partial D, \tag{1c}$$

$$\boldsymbol{\nu} \times \mu^{-1} \nabla \times \mathbf{E} - \boldsymbol{\nu} \times \nabla \times \mathbf{E}^s = \boldsymbol{\nu} \times \nabla \times \mathbf{E}^i \text{ on } \partial D, \tag{1d}$$

$$\lim_{r \to \infty} \left(\nabla \times \mathbf{E}^s \times \mathbf{x} - ikr \mathbf{E}^s \right) = 0. \tag{1e}$$

The domain D represents the support of the medium, which implies that $\epsilon = 1$ and $\mu = 1$ outside of D, and we assume that D is a Lipschitz domain with boundary ∂D and outward unit normal vector $\boldsymbol{\nu}$ and that $\mathbb{R}^3 \setminus \overline{D}$ is connected. The wave number k is given by $k = \sqrt{\epsilon_0 \mu_0}\omega$, where (ϵ_0, μ_0) are the material parameters for the homogeneous background material and ω is the frequency of the incident field. The Silver-Müller radiation condition (1e) is assumed to hold uniformly in all directions, and it follows that the scattered field $\mathbf{E}^s$ has the asymptotic expansion

$$\mathbf{E}^s(\mathbf{x}) = \frac{e^{ikr}}{r} \mathbf{E}_\infty(\hat{\mathbf{x}}) + O\left(\frac{1}{r^2}\right) \text{ as } r = |\mathbf{x}| \to \infty, \tag{2}$$

where $\mathbf{E}_\infty$ is the *far field pattern*. Our data consists of measurements of this far field pattern resulting from choosing $\mathbf{E}^i$ as a plane wave incident field with various incident directions $\mathbf{d} \in \mathbb{R}^3$, $|\mathbf{d}| = 1$, and polarizations $\mathbf{p} \in \mathbb{R}^3$, $\mathbf{p} \neq \mathbf{0}$, which we denote as $\mathbf{E}_\infty(\hat{\mathbf{x}}, \mathbf{d}; \mathbf{p})$.

In order to generate an eigenvalue problem, we compare this measured scattering data to the computed scattering data for an auxiliary problem. In the present work we consider the problem of finding a total field $\mathbf{E}_0$, a scattered field $\mathbf{E}_0^s$, and a scalar field P with $\int_B P\,dx = 0$ satisfying

$$\nabla \times \gamma^{-1}\nabla \times \mathbf{E}_0 - k^2\eta\mathbf{E}_0 + k^2\nabla P = \mathbf{0} \text{ in } B, \tag{3a}$$
$$\nabla \times \nabla \times \mathbf{E}_0^s - k^2\mathbf{E}_0^s = \mathbf{0} \text{ in } \mathbb{R}^3 \setminus \overline{B}, \tag{3b}$$
$$\nabla \cdot \mathbf{E}_0 = 0 \text{ in } B, \tag{3c}$$
$$\boldsymbol{\nu} \cdot \mathbf{E}_0 = 0 \text{ on } \partial B, \tag{3d}$$
$$\boldsymbol{\nu} \times \mathbf{E}_0 - \boldsymbol{\nu} \times \mathbf{E}_0^s = \boldsymbol{\nu} \times \mathbf{E}^i \text{ on } \partial B, \tag{3e}$$
$$\boldsymbol{\nu} \times \gamma^{-1}\nabla \times \mathbf{E}_0 - \boldsymbol{\nu} \times \nabla \times \mathbf{E}_0^s = \boldsymbol{\nu} \times \nabla \times \mathbf{E}^i \text{ on } \partial B, \tag{3f}$$
$$\lim_{r\to\infty} \left(\nabla \times \mathbf{E}_0^s \times \mathbf{x} - ikr\mathbf{E}_0^s\right) = 0, \tag{3g}$$

where B is chosen such that $D \subseteq B$, $\gamma \neq 1$ is a fixed parameter, and η is a complex number that will serve as the eigenparameter. By comparing the measured scattering data $\mathbf{E}(\hat{\mathbf{x}}, \mathbf{d}; \mathbf{p})$ to the auxiliary far field data $\mathbf{E}_{0,\infty}(\hat{\mathbf{x}}, \mathbf{d}; \mathbf{p})$ resulting from (3a)–(3g) for various plane wave incident fields, we generate the eigenvalue problem of finding a complex number η and nontrivial vector fields $\mathbf{w}$ and $\mathbf{v}$ and scalar field p with $\int_B p\,dx = 0$ satisfying

$$\nabla \times \mu^{-1}\nabla \times \mathbf{w} - k^2\epsilon\mathbf{w} = \mathbf{0} \text{ in } B, \tag{4a}$$
$$\nabla \times \gamma^{-1}\nabla \times \mathbf{v} - k^2\eta\mathbf{v} + k^2\nabla p = \mathbf{0} \text{ in } B, \tag{4b}$$
$$\nabla \cdot \mathbf{v} = 0 \text{ in } B, \tag{4c}$$
$$\boldsymbol{\nu} \cdot \mathbf{v} = 0 \text{ on } \partial B, \tag{4d}$$
$$\boldsymbol{\nu} \times (\mathbf{w} - \mathbf{v}) = \mathbf{0} \text{ on } \partial B, \tag{4e}$$
$$\boldsymbol{\nu} \times \left(\mu^{-1}\nabla \times \mathbf{w} - \gamma^{-1}\nabla \times \mathbf{v}\right) = \mathbf{0} \text{ on } \partial B. \tag{4f}$$

We call a value of η for which a nontrivial solution $(\mathbf{w}, \mathbf{v}, p)$ of (4a)–(4f) exists a *modified transmission eigenvalue*, and in the following theorem we summarize some basic properties of these eigenvalues [3].

Theorem 1. *If $\gamma \neq 1$, then the set of modified transmission eigenvalues is discrete without finite accumulation point. If μ and ϵ are real-valued, then all of the eigenvalues are real and infinitely many exist.*

III. Detection of Eigenvalues

For practical applications, it is necessary that modified transmission eigenvalues can be computed from measured scattering data $\mathbf{E}_\infty$ and computed auxiliary data $\mathbf{E}_{0,\infty}$. We first define the *modified far field operator* $\boldsymbol{\mathcal{F}}$ as the integral operator

$$(\boldsymbol{\mathcal{F}}\mathbf{g})(\hat{\mathbf{x}}) := \int_{\mathbb{S}^2} \left[\mathbf{E}_\infty(\hat{\mathbf{x}}, \mathbf{d}; \mathbf{g}(\mathbf{d})) - \mathbf{E}_{0,\infty}(\hat{\mathbf{x}}, \mathbf{d}; \mathbf{g}(\mathbf{d}))\right] ds(\mathbf{d}), \tag{5}$$

where $\mathbf{g}$ is a function defined on the unit sphere $\mathbb{S}^2$. With the far field pattern of an *electric dipole* with polarization $\mathbf{q}$ given by

$$\mathbf{E}_{e,\infty}(\hat{\mathbf{x}}, \mathbf{z}; \mathbf{q}) := \frac{ik}{4\pi}(\hat{\mathbf{x}} \times \mathbf{q}) \times \hat{\mathbf{x}}e^{-ik\hat{\mathbf{x}}\cdot\mathbf{z}}, \ \hat{\mathbf{x}} \in \mathbb{S}^2, \tag{6}$$

and the *Herglotz wave function* defined by

$$\mathbf{v}_\mathbf{g}^i(\mathbf{x}) := ik\int_{\mathbb{S}^2} e^{-ik\mathbf{x}\cdot\mathbf{d}}\mathbf{g}(\mathbf{d})\,ds(\mathbf{d}), \ \mathbf{x} \in \mathbb{R}^3, \tag{7}$$

we state the following results which allow for the detection of eigenvalues from measured scattering data through the linear sampling method [3].

Theorem 2. *(i) Let $\mathbf{z} \in B$. If η is not a modified transmission eigenvalue, then for every $\delta > 0$ there exists $\mathbf{g}_\mathbf{z}^\delta$ satisfying*

$$\lim_{\delta\to 0}\|\boldsymbol{\mathcal{F}}\mathbf{g}_\mathbf{z}^\delta - \mathbf{E}_{e,\infty}(\hat{\mathbf{x}}, \mathbf{z}; \mathbf{q})\|_{\mathbf{L}_t^2(\mathbb{S}^2)} = 0 \tag{8}$$

such that the sequence $\{\|\mathbf{v}_{\mathbf{g}_\mathbf{z}^\delta}^i\|_{\mathbf{H}(\mathbf{curl},B)}\}_{\delta>0}$ is bounded.

(ii) If η is a modified transmission eigenvalue and the sequence $\{\mathbf{g}_\mathbf{z}^\delta\}$ satisfies (8), then the sequence $\{\|\mathbf{v}_{\mathbf{g}_\mathbf{z}^\delta}^i\|_{\mathbf{H}(\mathbf{curl},B)}\}_{\delta>0}$ cannot be bounded for almost every $\mathbf{z} \in B_\rho$, where $B_\rho \subset B$ is an arbitrary ball of radius $\rho > 0$.

We may use Theorem 2 to compute eigenvalues in the following manner. For various values of η in a region containing an eigenvalue , we compute the auxiliary far field data $\mathbf{E}_{0,\infty}(\hat{\mathbf{x}}, \mathbf{d}; \mathbf{p})$ and construct an approximate solution $\mathbf{g} = \mathbf{g}_{\mathbf{z},\eta}$ to a regularized version of the modified far field equation $\boldsymbol{\mathcal{F}}\mathbf{g} = \mathbf{E}_{e,\infty}(\cdot, \mathbf{z}; \mathbf{q})$. We search for eigenvalues as peaks in the graph of the map $\eta \mapsto \|\mathbf{g}_{\mathbf{z},\eta}\|_{\mathbf{L}_t^2(\mathbb{S}^2)}$. We refer to [4] for the same approach used to compute electromagnetic Stekloff eigenvalues from far field data.

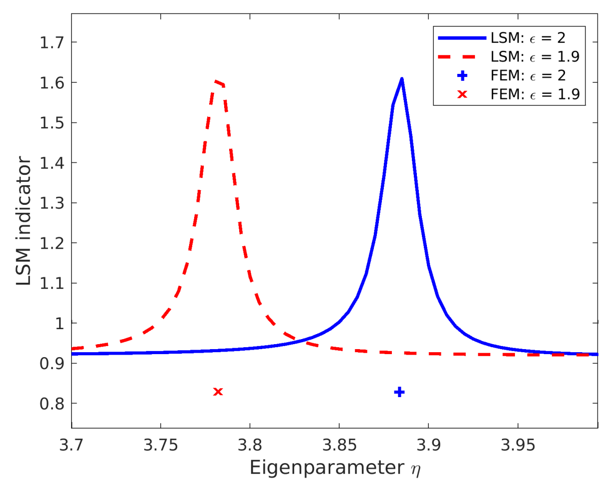

Fig. 1. A shift in an eigenvalue due to an overall change in a constant electric permittivity with $\mu = 1$. The simulated electric far field data corresponds to the unit sphere with $k = 2$ and 2% noise.

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