2.3. Continuity

Def: A function f is continuos at a point $x=c$ if:

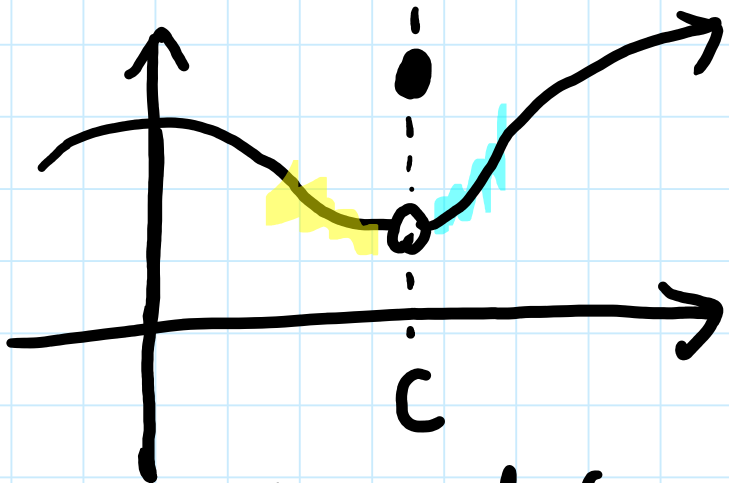
1) $f(c)$ defined

2) $\lim\limits_{x\to c} f(x)$ exists

3) $f(c) = \lim\limits_{x\to c} f(x)$

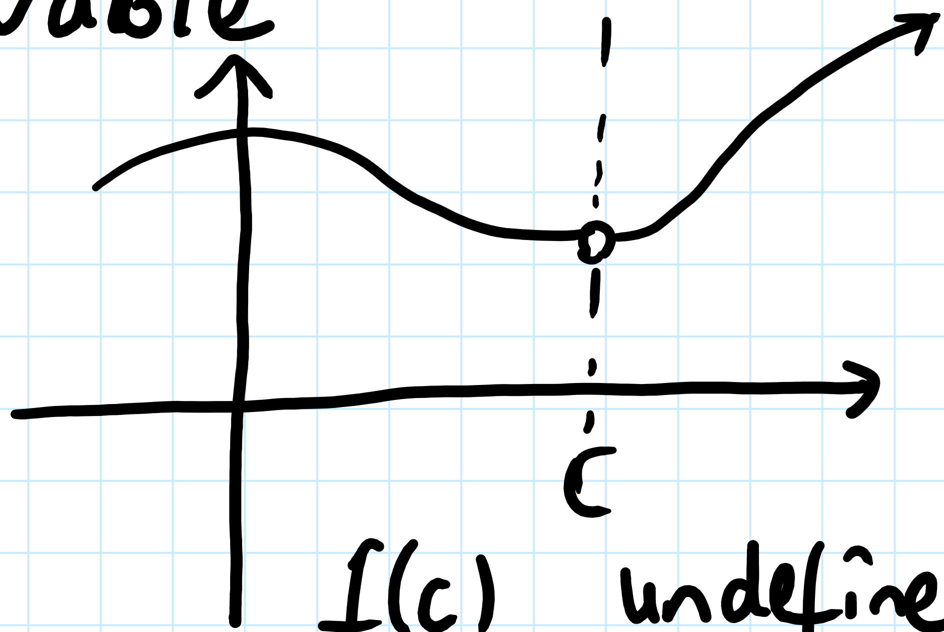
4 types of discontinuity

1) Hole / Removable



$f(c)$ defined

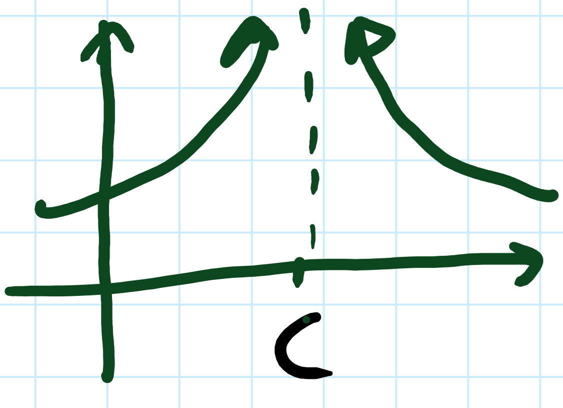
$\lim\limits_{x\to c} f(x)$ exists



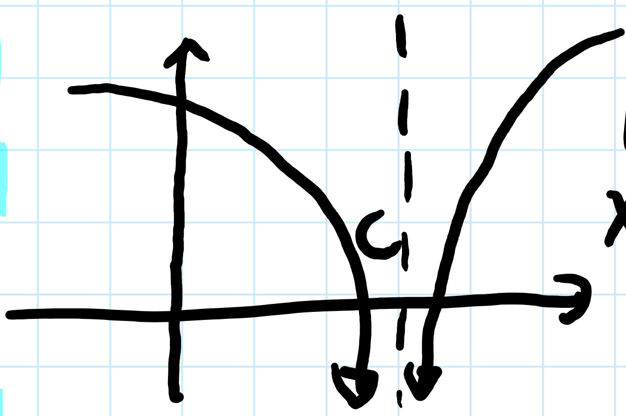
$f(c)$ undefined

$\lim\limits_{x\to c} f(x)$ exists

2) Pole / Infinite

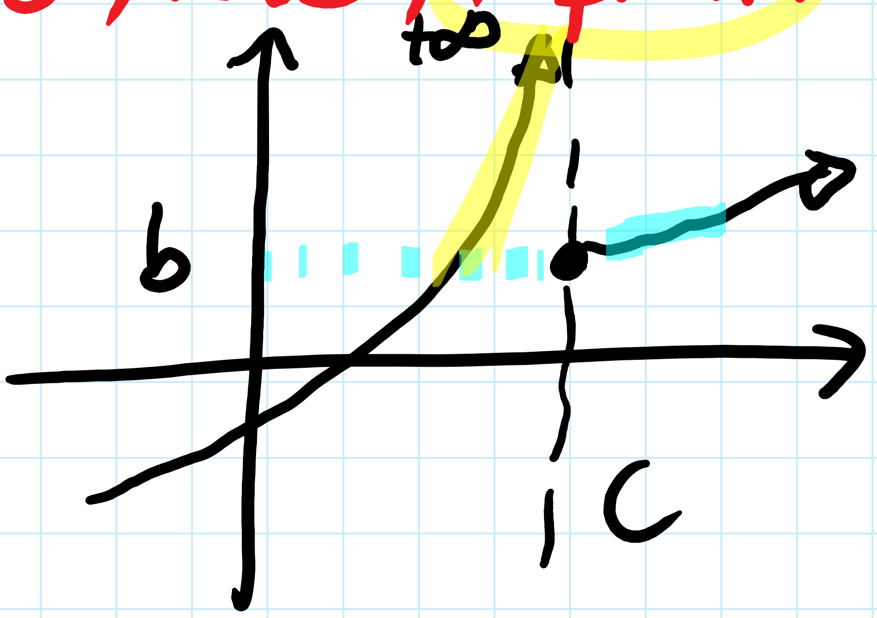


$\lim\limits_{x\to c} f(x) = +\infty$



$\lim\limits_{x\to c} f(x) = -\infty$

2) Pole / Infinite



$f(c)$ defined

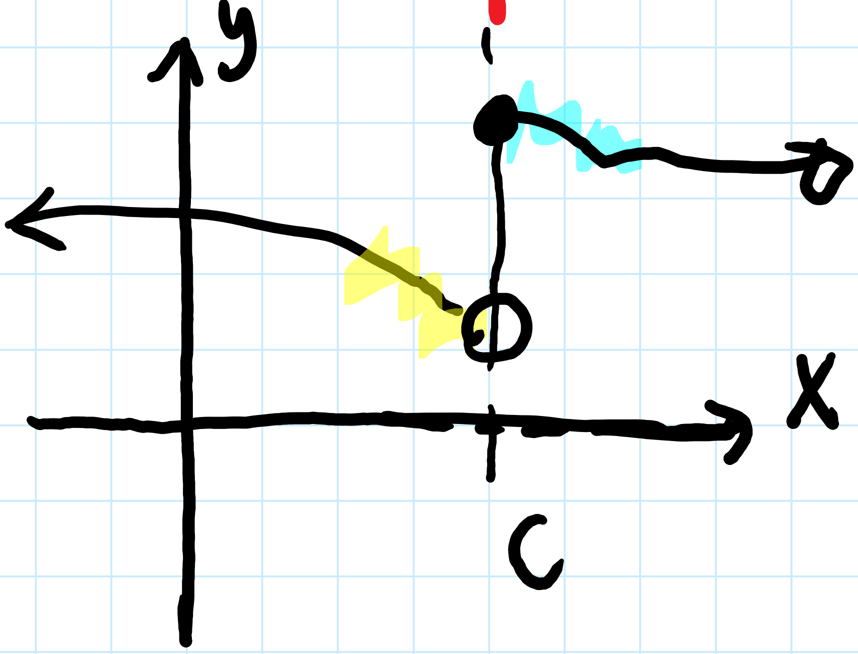
continues...

$$\lim_{x\to c^-} f(x) = +\infty$$

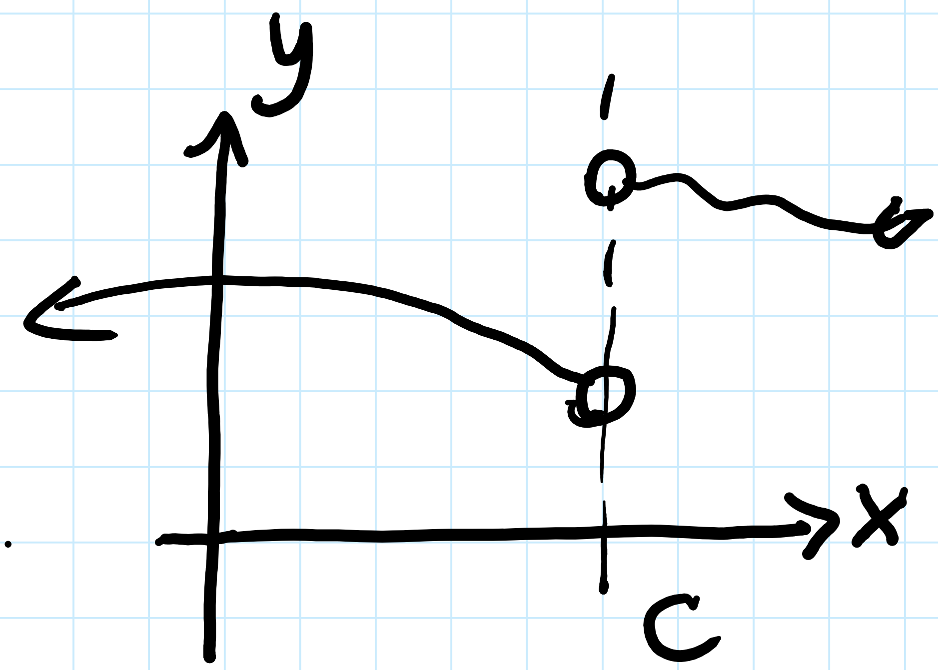
$$\lim_{x\to c^+} f(x) = b$$

$$\underbrace{\phantom{\lim_{x\to c^+} f(x) = b}}_{\lim_{x\to c} f(x) \text{ DNE}}$$

3) Jump

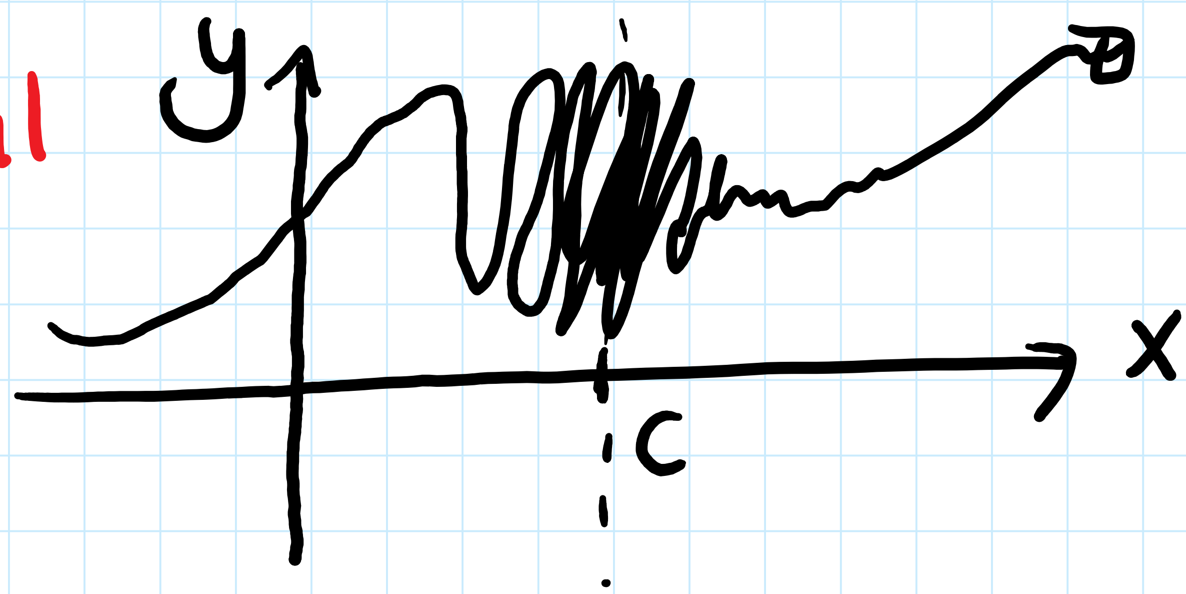


$f(c)$ defined



$f(c)$ undefined

4) Essential



Test Continuity of given functions at $x=1$

Continuity Test:

1) $f(1)$ is def.

2) $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x)$

$\lim_{x\to1} f(x)$ exists

3) Check ① & ②

1) $f(x) = \dfrac{x^2+2x-3}{x-1}$

$f(1)$ is undef: $f(1) = \dfrac{1^2+2-3}{1-1} = "\dfrac{0}{0}"$

Hole type of discontinuity

2) $g(x) = \dfrac{x^2+2x-3}{x-1}$ if $x \neq 1$; $g(x) = 4$ if $x=1$

Solution:

1) $g(1) = 4$

2) $\lim_{x\to1} g(x)$ exist? try "D.S.P" $"\dfrac{0}{0}"$

$$\lim_{x\to1} g(x) = \lim_{x\to1} \frac{x^2+2x-3}{x-1} = \lim_{x\to1} \frac{\cancel{(x-1)}(x+3)}{\cancel{x-1}}$$

$$\lim_{x\to1} (x+3) \overset{\text{"D.S.P"}}{=} 1+3 = 4$$

3) check if $g(1) = \lim_{x\to1} g(x) \Rightarrow 4=4$ ✓

therefore; $g(x)$ is continuous at $x=1$

3) $h(x) = \dfrac{x^2+2x-3}{x-1}$ If $x \neq 1$; $h(x)=6$ if $x=1$

Solution:

1) $h(1) = 6$ "D.S.P"

2) $\lim\limits_{x\to 1} h(x) = \lim\limits_{x \to 1} (x+3) = 4$

3) check $h(1) \stackrel{?}{=} \lim\limits_{x\to 1} h(x)$

$$6 \neq 4$$

Therefore; (since the func. failed the 3rd test)
$h(x)$ is not continuous at $x=1$

Exp) $F(x) = \dfrac{x+3}{x-1}$ if $x \neq 1$; $F(x) = 4$ if $x = 1$

solution:

1) $F(1) = 4$

2) $\lim\limits_{x \to 1} F(x)$

$\lim\limits_{x \to 1^-} F(x)$ | $\lim\limits_{x \to 1^+} F(x)$

$\lim\limits_{x \to 1^-} \left(\dfrac{x+3}{x-1}\right) = -\infty$ | $\lim\limits_{x \to 1^+} \left(\dfrac{x+3}{x-1}\right) = +\infty$

$\lim\limits_{x \to 1^-} F(x) \neq \lim\limits_{x \to 1^+} F(x)$ } therefore $\lim\limits_{x \to 1} F(x)$ DNE

The function $F(x)$ is not continuous at $x = 1$.

Q: Are there points that we should be "suspicious" of that the discontinuity can occur?

Yes!

1) The defining rule for the function changes (transition points of piecewise functions)

2) Substitution of $x=c$ causes division by 0 in the function (rational functions)

Exp) Find the interval on which $f(x)$

$f(x) = \frac{x^2-1}{x^2-4}$, is continuous?

Solution: $f(x)$ is not defined when $x^2-4=0$; $x=2$ or $x=-2$

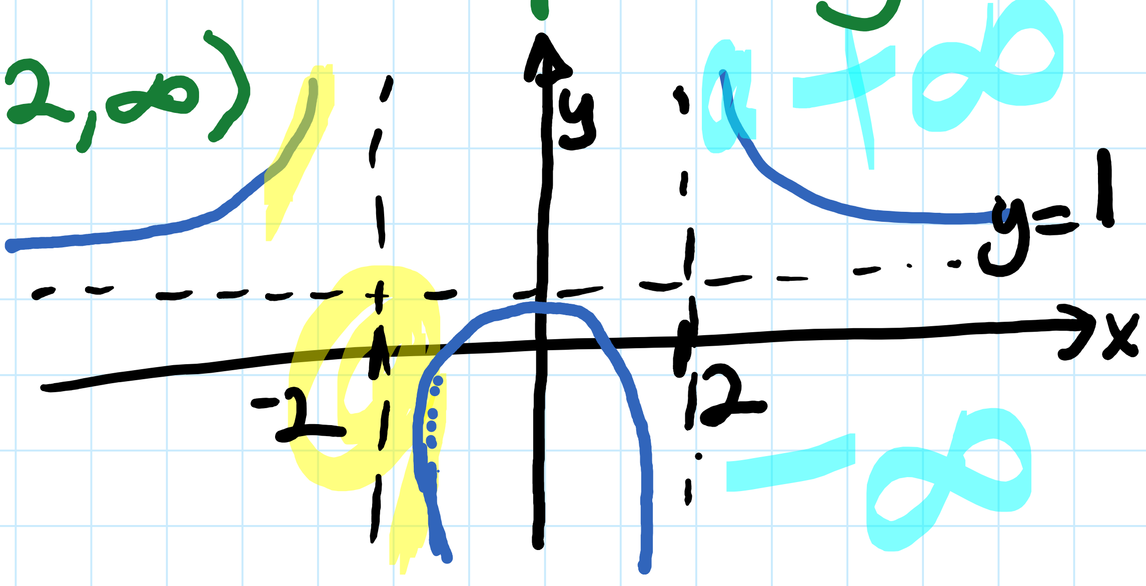
$f(x)$ is continuous on

$(-\infty,-2) \cup (-2,2) \cup (2,\infty)$

Graphically

V.A: $x = \mp 2$

H.A: $y=1$



Exp) Check continuity in the given interval:

Let

$$f(x) = \begin{cases} 3-x, & \text{if } -5 \leq x < 2 \\ x-2, & \text{if } 2 \leq x < 5 \end{cases}$$

① ②

Solution:

transition point $x=2$

$[-5, 5)$

equal to include

not eq. to exclude

1) $f(2) = 2-2 = 0$

2) $\lim\limits_{x \to 2} f(x) = ?$

"D.S.P."

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (3-x) = 3-2 = 1$

"D.S.P."

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x-2) = 2-2 = 0$

Since: $\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x)$ $[1 \neq 0]$

$\lim\limits_{x \to 2} f(x)$ DNE

$f(x)$ is discontinuous at $x=2$.

Thus, $f(x)$ is continuous in the interval of $[-5, 2)$ and $(2, 5)$.

Exp) Find the interval on which $g(x)$ is continuous.

Let

$$g(x) = \begin{cases} 2-x, & \text{if } -5 \le x < 2 \quad (1) \\ x-2, & \text{if } 2 \le x < 5 \quad (2) \end{cases}$$

Solution: function changes rule at $x=2$.

1) $g(2) = 2-2 = 0$

"D.S.P."

2) $\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (2-x) = 2-2 = 0$

$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (x-2) = 2-2 = 0$

$\lim_{x \to 2} g(x) = 0$

3) $g(2) = \lim_{x \to 2} g(x)$ $\quad [0 = 0]$

Therefore: function $g(x)$ is continuous throughout the interval $[-5, 5)$.

Exp) $f(x) = \begin{cases} x+a, & \text{if } x<0 \quad ① \\ 5, & \text{if } x=0 \quad ② \\ \dfrac{\sin(bx)}{x}, & \text{if } x>0 \quad ③ \end{cases}$

Find the values of a, b that makes f continuous at x=0, or show that no such values exist.

Solution: 1) $f(0)=5$

"D.S.P."

2) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+a) = 0+a = a$

Recall: $\lim_{x\to 0} \frac{\sin(\theta x)}{\theta x} = 1$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin(bx)}{x}$$

$$= \lim_{x\to 0^+} \left(\frac{\sin(bx)}{b\cdot x}\cdot b\right)$$

$$= \lim_{x\to 0^+} \left(\frac{\sin(bx)}{bx}\right)\cdot \lim_{x\to 0^+}(b)$$

$$= b$$

In order for $f(x)$ to be continuous at $x=0$;

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$a = b = 5$$

Exp) Find constants a, b so that the function will be continuous for all x.

Q45 of Text.

$f(2)+3=f(0)$ (Given)

$$f(x)=\begin{cases} ax+b, & \text{if } x>1 \quad ① \\ 3, & \text{if } x=1 \quad ② \\ x^2-4x+b+3, & \text{if } x<1 \quad ③ \end{cases}$$

Solution:

$$f(2)+3=f(0)$$

$$a\cdot 2+\cancel{b}+\cancel{3}=0^2-4\cdot 0+b+3$$

$$\cancel{-b}\ \cancel{-3}\qquad -b-3$$

$$2a=0$$

$$a=0$$

Test the continuity of f(x) at $x=1$ (where the function rule changes, "transition point")

① $f(1)=3$

② $\lim_{x\to 1} f(x)$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x^2-4x+b+3)=1^2-4\cdot 1+b+3$$
$$=b$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(ax+b)=a\cdot 1+b=a+b$$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)$$
$$b=a+b \quad (a=0)$$
$$b=b \checkmark$$

③ For f(x) to be continuous at $x=1$;

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)$$
$$b=b=3$$

$b=3$
$a=0$