2.2. Algebraic Computation Of Limits

In 2.1; we learned about finding limits numerically and graphically;

In 2.2; we will learn about finding limits algebraically (relying on the direct substitution method first, if it doesn't work; we will use factoring, trig. identities, rationalizing.

"Direct Substitution Property" → DSP

If $\lim_{x \to c} f(x) = f(c)$; then f has the DSP at $x=c$ (as long as c is in the domain of $f(x)$)

E.g.: $\lim_{x \to 2} f(x) = \lim_{x \to 2} (2x^5 - 9x^3 + 3x^2 - 11) = ?$

Solution:

"DSP" $\Rightarrow$ $2 \cdot 2^5 - 9 \cdot 2^3 + 3 \cdot 2^2 - 11$

$2 \cdot 32 - 9 \cdot 8 + 3 \cdot 4 - 11$

$\underbrace{64 - 72}_{} + \underbrace{12 - 11}_{} = -8 + 1 = \boxed{-7}$

E.g. Compute $\lim_{z \to -1} \dfrac{z^3-3z+7}{5z^2+9z+6}$

Solution: check if -1 is in domain of the rational function.

$5z^2+9z+6$; $z=-1 \Rightarrow 5(-1)^2+9(-1)+6$

$5-9+6=2$

lim notation is important

Use DSP for the numerator:

$$\lim_{z \to -1} \frac{z^3-3z+7}{5z^2+9z+6} = \frac{(-1)^3-3(-1)+7}{2} = \frac{-1+3+7}{2} = \boxed{\frac{9}{2}}$$

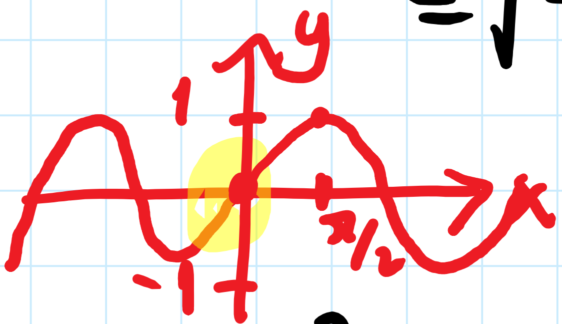
Exp) Power (or Root) Function

Compute $\lim_{x\to -2} f(x) = \lim_{x\to -2} \sqrt[3]{x^2-3x-2}$

Solution: DSP: $\lim_{x\to -2} \sqrt[3]{x^2-3x-2} = \sqrt[3]{(-2)^2-3(-2)-2}$

$= \sqrt[3]{4+6-2} = \sqrt[3]{8} = 2$

Exp) Trig. Functions

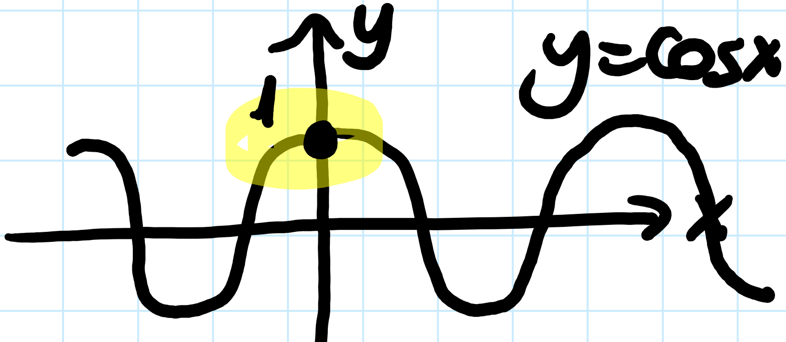


graph of y = Sin x

Compute

a) $\lim_{x\to 0} (\sin x)^2 = \left[\lim_{x\to 0} \sin x\right]^2 = 0^2 = 0$

Exp) Compute $\lim_{x\to 0}(1-\cos x)$



Solution: "DSP"

$$\lim_{x\to 0}(1-\cos x) = 1-\cos 0 = 1-1 = \boxed{0}$$

Exp) Compute $\lim_{x\to 1}(x^2\cdot\cos\pi x)$.

Solution:

$$\left(\lim_{x\to 1} x^2\right)\cdot\left(\lim_{x\to 1}\cos\pi x\right)$$

"DSP" "DSP"

$$= 1^2\cdot\cos(\pi\cdot 1) = 1\cdot\cos\pi = -1$$

What to do when division by 0 after DSP?

E.g.: $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2+x-6}{x-2} = ?$ try "DSP" $\frac{2^2+2-6}{2-2} = \text{"}\frac{0}{0}\text{"}$

Try Factoring.

$$\lim_{x\to 2} \frac{\cancel{(x-2)}(x+3)}{\cancel{x-2}} = \lim_{x\to 2} (x+3) \quad [x \neq 2]$$

"DSP"

$$= 2+3 = \boxed{5}$$

Eg: $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = ?$

try "D.S.P"

$\frac{\sqrt{4}-2}{4-4} = \text{"}\frac{0}{0}\text{"}$

$a^2-b^2=(a-b)(a+b)$

$(\sqrt{x})^2-2^2=(\sqrt{x}-2)\cdot(\sqrt{x}+2)$

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}$$

$(\sqrt{x}-2)\cdot(\sqrt{x}+2)=(\sqrt{x})^2-2^2$

$=x-4$

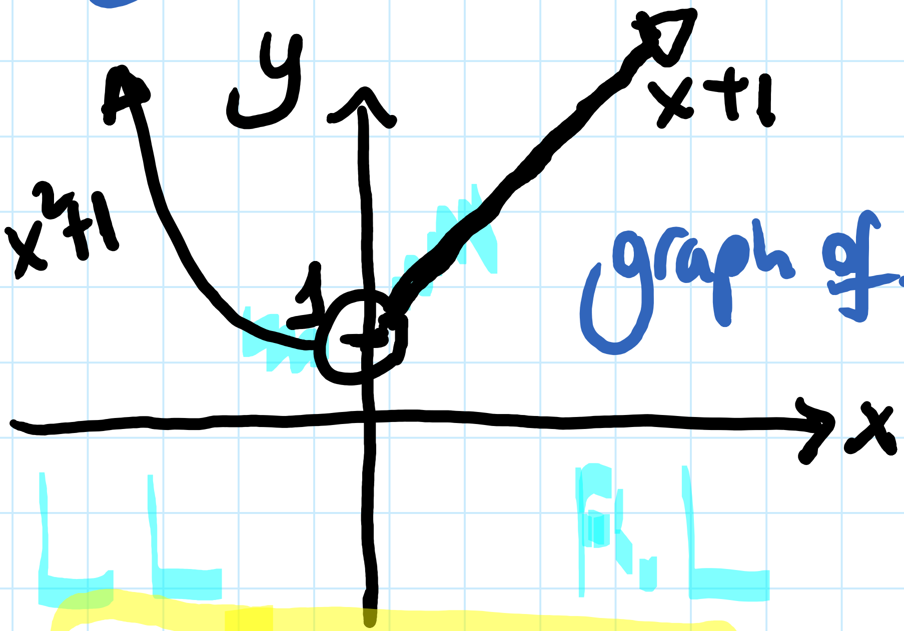
$$=\lim_{x\to 4} \frac{\cancel{x-4}^{\,1}}{\cancel{(x-4)}\cdot(\sqrt{x}+2)}$$

$$=\lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \boxed{\frac{1}{4}}$$

Limits of Piecewise-Defined Functions

Compute $\lim_{x\to 0} f(x)$ where $f(x)=\begin{cases} x+1, & \text{if } x>0 \\ x^2+1, & \text{if } x<0 \end{cases}$

Solution:



graph of f(x)

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2+1) = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+1) = 1$$

$$\lim_{x\to 0^+} (x+1) = \lim_{x\to 0^-} (x^2+1) = 1$$

$$\boxed{\lim_{x\to 0} f(x) = 1}$$

E.g: Compute $\lim_{x\to 0} g(x)$ where $g(x) = \begin{cases} x+5, & \text{if } x>0 \\ x, & \text{if } \underbrace{x<0} \end{cases}$

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x = 0$$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (x+5) = 5$$

$0 \neq 5$

$x=0$ transition point

"$g(x)$ transitions from one piece to the other piece"

Answer: $\lim_{x\to 0} g(x)$ DNE

because $\lim_{x\to 0^-} g(x) \neq \lim_{x\to 0^+} g(x)$

Special Trig. Limits to Memorize

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \; ; \; \lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1$$

Exp $\lim_{x \to 0} \frac{\sin 3x}{5x} = ?$

$$\lim_{x \to 0} \frac{\sin 3x}{5x} \cdot \frac{3x}{3x} = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \frac{3\cancel{x}}{5\cancel{x}}$$

$$\frac{\sin 3x}{3x} \to 1$$

$$= \lim_{x \to 0} \frac{3}{5} = \boxed{\frac{3}{5}}$$

Exp) Calculate $\lim_{x\to 0} \frac{\tan(8x)}{\sin(3x)}$.

Recall: $\tan\theta = \frac{\sin\theta}{\cos\theta}$

Solution.

$$\lim_{x\to 0} \frac{\frac{\sin(8x)}{\cos(8x)}}{\sin(3x)} = \lim_{x\to 0} \frac{\sin(8x)}{\cos(8x)} \cdot \frac{1}{\sin(3x)}$$

$$= \lim_{x\to 0} \frac{\sin(8x)}{\cos(8x)} \cdot \frac{8x}{8x} \cdot \frac{1}{\sin(3x)} \cdot \frac{3x}{3x}$$

1 ← 1 →

$$= \lim_{x\to 0} \frac{\sin(8x)}{8x} \cdot \frac{8x}{\cos(8x)} \cdot \frac{3x}{\sin 3x} \cdot \frac{1}{3x}$$

"DSP"

$$= \lim_{x\to 0} \frac{8\cancel{x}}{\cos(8x)\cdot 3\cancel{x}} = \lim_{x\to 0} \frac{8}{3\cdot\cos(8x)} = \frac{8}{3\cdot\cos 0} = \frac{8}{3}$$

E.g: Compute $\lim_{x \to 1} \left(\frac{\frac{1}{x} - 1}{x-1} \right)$

Solution:

try "DSP" $\Rightarrow$ "$\frac{0}{0}$"

$$\lim_{x\to 1}\left(\frac{\frac{1}{x}-\frac{1}{1}\frac{(x)}{(x)}}{x-1}\right) = \lim_{x\to 1}\left(\frac{\frac{1-x}{x}}{\frac{x-1}{1}}\right) = \lim_{x\to 1}\left(\underbrace{\frac{\cancel{1-x}^{\,-1}}{x}\cdot\frac{1}{\cancel{x-1}}}_{1-x = -(x-1)}\right)$$

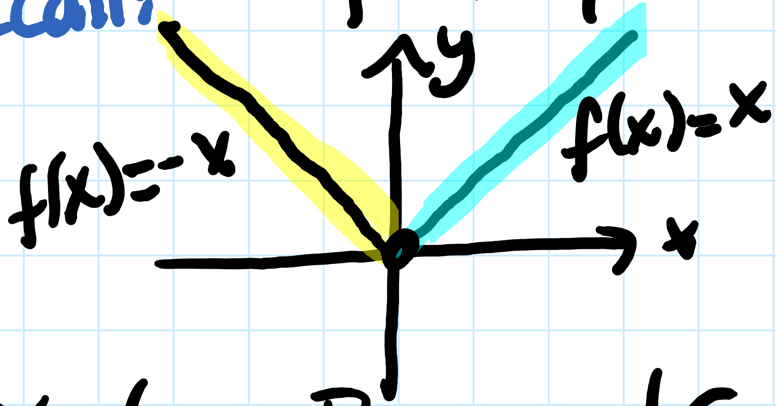
$$= \lim_{x\to 1}\left(\frac{-1}{x}\right) \overset{\text{"DSP"}}{=} \frac{-1}{1} = \boxed{-1}$$

Finding the Limits of Absolute Value Functions

Exp) compute $\lim\limits_{x\to 6} \frac{|x-6|}{x-6}$

Solution:

Recall: $f(x)=|x|$



$$|x| = \begin{cases} -x, & \text{if } x<0 \\ x, & \text{if } x\geq 0\end{cases}$$

$x=6$ is a transition p. for $\frac{|x-6|}{x-6}$

LL

$$\lim_{x\to 6^-} \frac{-(x-6)}{x-6} = \lim_{x\to 6^-} -1 = -1$$

RL

$$\lim_{x\to 6^+} \frac{(x-6)}{x-6} = \lim_{x\to 6^+} 1 = 1$$

LL $\neq$ RL $(-1\neq 1)$ therefore: $\lim\limits_{x\to 6}\frac{|x-6|}{x-6}$ $\boxed{\text{DNE}}$

Exp) Compute $\displaystyle\lim_{x\to 0} \frac{(x+3)^2-9}{x}$

Solution:

try "D.S.P" $\displaystyle\frac{(0+3)^2-9}{0} = \frac{9-9}{0} = \text{"}\frac{0}{0}\text{"}$

$$\lim_{x\to 0} \frac{(x+3)^2-3^2}{x} = \lim_{x\to 0} \frac{(x+3-3)(x+3+3)}{x}$$

$$= \lim_{x\to 0} \frac{\cancel{x}\cdot(x+6)}{\cancel{x}} = \lim_{x\to 0} x+6 = 0+6 = 6$$

"DSP"

OR $\displaystyle\lim_{x\to 0} \frac{x^2+6x+9-9}{x} = \lim_{x\to 0} \frac{\cancel{x}(x+6)}{\cancel{x}} = 6$