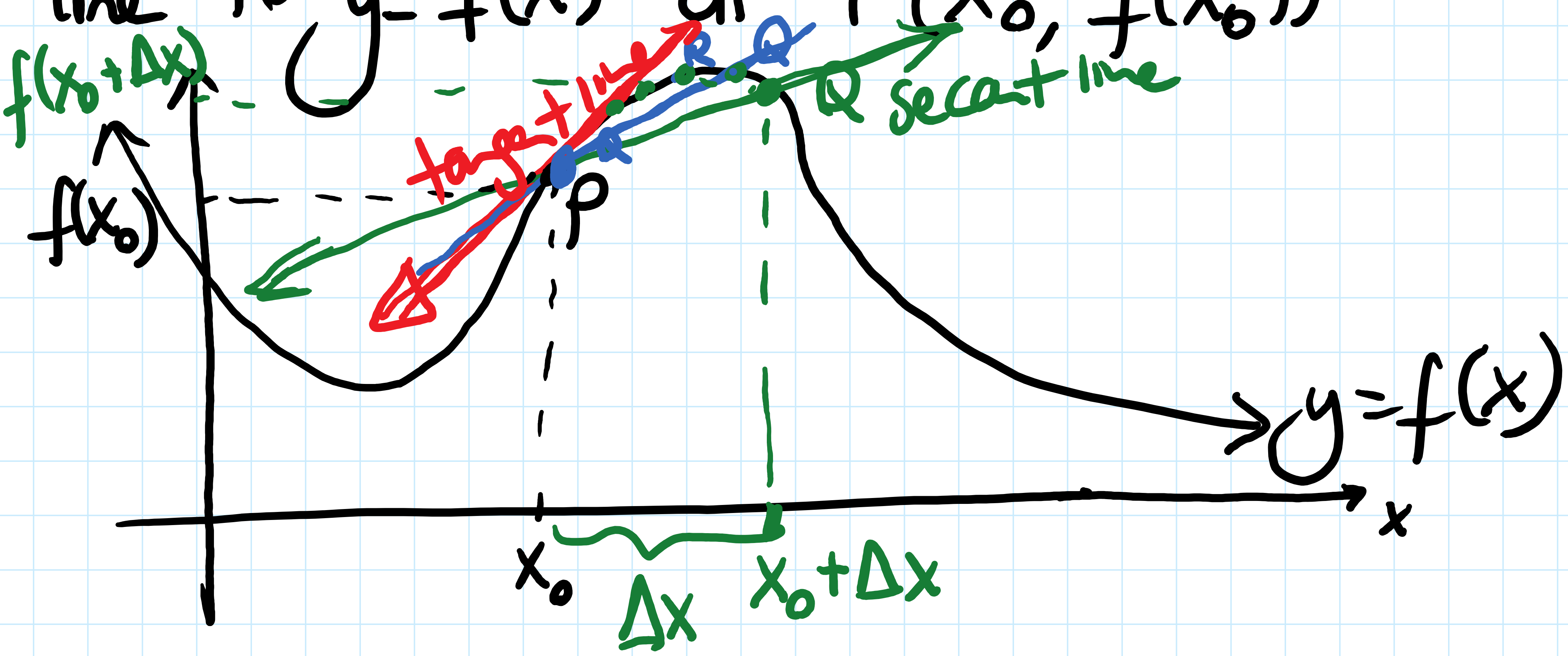
# Chapter 3. Differentiation

## 3.1. Introduction to Derivative: Tangents

**Goal:** To find the slope of the tangent line to $y=f(x)$ at $P(x_0, f(x_0))$




**Strategy** is to approximate the tangent line by other lines whose slopes can be computed directly.

**Secant Line:** The line joining the given point $P$ to the neighboring point $Q$ on the graph of $f$.

$m_{sec}$ between $P(x_0, f(x_0))$ & $Q(x_0+\Delta x, f(x_0+\Delta x))$

$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{f(x_0+\Delta x) - f(x_0)}{\cancel{x_0}+\Delta x - \cancel{x_0}}$$

$$m_{sec} = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

$P(a, f(a))$

$P(x_0, f(x_0))$

$$m_{tan} = \lim_{\Delta x \to 0} \boxed{\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}}$$

$m_{sec}$

$$m_{tan} = \lim_{\Delta x \to 0} (m_{sec})$$

Exp) Derive a formula for the slope of the tangent line to the graph of $f(x)=x^2$.

$x_0 \to x$

<u>Solution:</u>

$$m_{tan} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$f(x)=x^2$

$f(x+\Delta x)=(x+\Delta x)^2$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2}+2x\cdot\Delta x+(\Delta x)^2 \cancel{-x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\cdot\Delta x+(\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x+\Delta x)}{\cancel{\Delta x}}$$

$$\boxed{m_{tan} = 2x}$$

Find the equation of a tangent line to $f(x)=x^2$ at $P(2,4)$.

$x=2$
$f(2)=2^2=4$

$$\boxed{\begin{array}{l} m_{tan}=2x \text{ at } (2,4) \\ 2\cdot 2=4 \end{array}}$$

$m=4$

$P(2,4)$

$P(x_1,y_1)$

$y-y_1=m(x-x_1)$

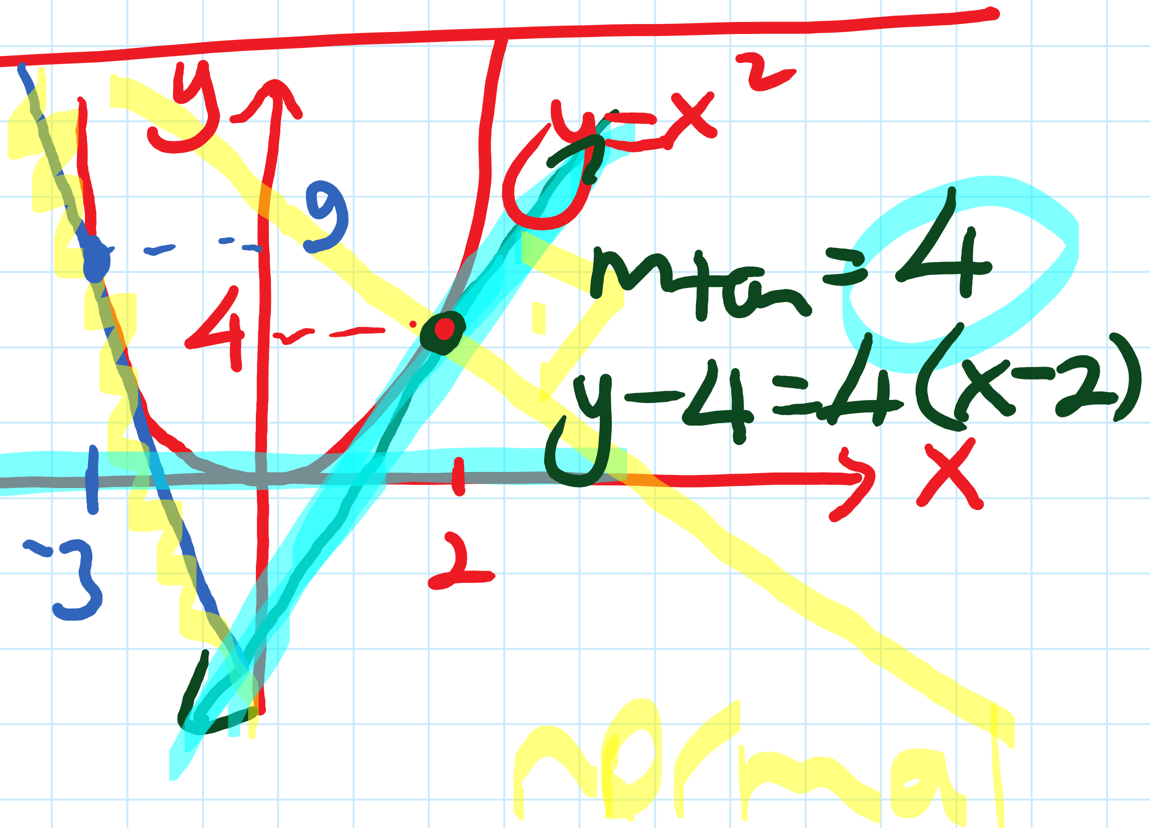
$$\boxed{y-4=4(x-2)}$$

$R(-3,9)$

$m_{tan}=2x$

$m_{tan}$ at $R$: $2(-3)$

$-6$




For different points on $f(x)$:
the value of $m_{tan}$ changes.
$m_{tan}$ at point R: $-6$
$m_{tan}$ at point P: $4$

**Derivative:** Limit of difference quotient gives a formula for the slope of the tangent line to the graph of $f(x)$ at $P(x_0, f(x_0))$ is called the derivative of $f'(x)$.

To differentiate a function $f$ at $x_0$ means to find the derivative at $(x=x_0)$

**Exp)** Differentiate $f(x)=\sqrt{x}$

Solution: $\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

$f(x)=\sqrt{x}$

$f(x+\Delta x)=\sqrt{x+\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\left(\sqrt{x+\Delta x}-\sqrt{x}\right)}{\Delta x} \cdot \frac{\left(\sqrt{x+\Delta x}+\sqrt{x}\right)}{\left(\sqrt{x+\Delta x}+\sqrt{x}\right)}$$

$a^2-b^2=(a-b)(a+b)$

$x+\Delta x-x=\left(\sqrt{x+\Delta x}-\sqrt{x}\right)\left(\sqrt{x+\Delta x}+\sqrt{x}\right)$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}\left(\sqrt{x+\Delta x}+\sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

**Recall:** $f(x) = \sqrt{x}$ $\qquad$ $f'(x) = \dfrac{1}{2\sqrt{x}}$

domain of $f(x)$: $x \geq 0$

$[0, \infty)$

domain of $f'(x)$: $x > 0$

$(0, \infty)$

$f(x)$ is def. in $[0, \infty)$

$f(x)$ is differentiable only in $(0, \infty)$

A function does not need to be differentiable at every point in its interval.

Exp) Differentiate $f(x) = \frac{1}{x}$

Solution: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{x}{x(x+\Delta x)} - \frac{x+\Delta x}{x(x+\Delta x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{x-(x+\Delta x)}{x(x+\Delta x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{\cancel{x}-\cancel{x}-\Delta x}{x(x+\Delta x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{-\cancel{\Delta x}}{x(x+\Delta x)}}{\cancel{\Delta x}} = \lim_{\Delta x \to 0} \frac{-1}{x(x+\Delta x)}$$

$$= \frac{-1}{x(x+0)} = \frac{-1}{x^2} = m_{ta}$$

Eq. of the tangent line to $f(x)$ at $x=2$

$m_{tan} = \frac{-1}{x^2}$; $m_{tan} = \frac{-1}{x^2}\Big|_{x=2} = \frac{-1}{2^2} = \frac{-1}{4}$

If $x=2$; $f(x) = \frac{1}{x}$; $f(2) = \frac{1}{2}$

$y - \frac{1}{2} = \frac{-1}{4}(x-2)$

Find the equation of the normal line to $f(x)$ at a given point.

$m_{tan}$ at $x=2$; $f'(2) = \frac{-1}{4}$

$m_{normal} = \frac{-1}{m_{tan}}$ $\Big\}$ $m_{normal} = \frac{-1}{\frac{-1}{4}} = 4$

$x=2$, $f(x) = \frac{1}{x}$ ; $f(2) = \frac{1}{2}$

$P(2, \frac{1}{2})$
$(x_1, y_1)$

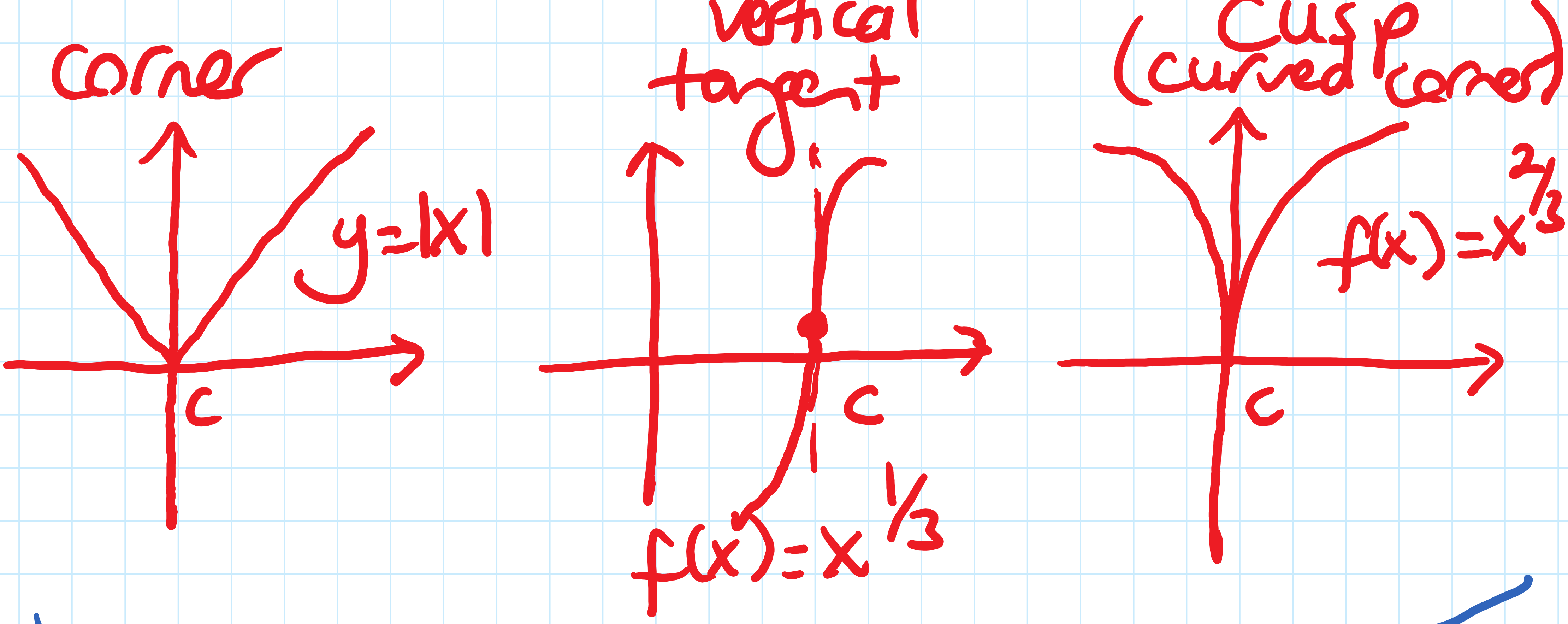
$m_{normal} = 4$

$y - y_1 = m(x - x_1)$

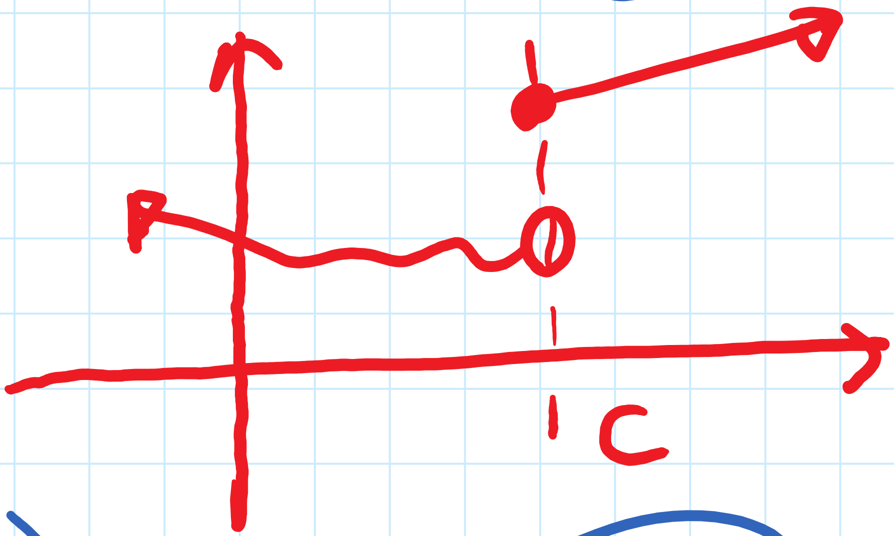
$y - \frac{1}{2} = 4(x-2)$

# Existence of Derivatives

Common ways for a derivative to fail to exist at point $P(c, f(c))$ in the domain of $f(x)$




Continuous function




point of discontinuity at $x = c$

discontinuous function

A function is differentiable only if the limit exists.

Exp) Show that the absolute value function $f(x)=|x|$ is not differentiable at $x=0$.

Solution:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Eval. ↑ at $x=0$

$$\lim_{\Delta x \to 0} \frac{f(\Delta x)-0}{\Delta x}$$

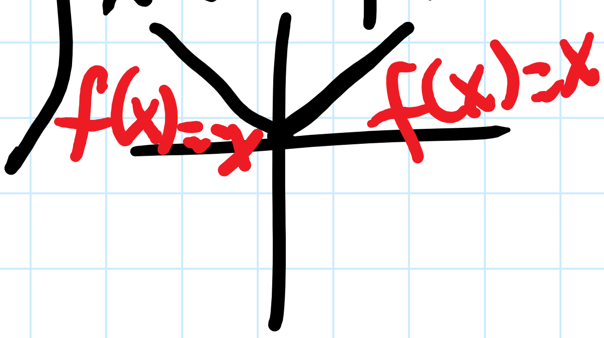
$$\lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}$$

$f(x)=|x|$

$f(0)=0$

$f(\Delta x)=|\Delta x|$

$f(x)=|x| \begin{cases} x \geq 0 & f(x)=x \\ x<0 & f(x)=-x \end{cases}$




$$\lim_{\Delta x \to 0^-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1$$

$$\lim_{\Delta x \to 0^+} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = 1$$

$-1 \neq 1$

The left-hand and right-hand limits are NOT the same; therefore the limit DNE.

The derivative doesn't exist at $x=0$.

# Differentiability Implies Continuity

1) If a function is differentiable at $x=c$, Then it must be continuous at $x=c$.

2) If a function is continuous at $x=c$, Then it may or may not differentiable at $x=c$.

3) If a function is discontinuous at $x=c$, Then it can't possibly have a derivative at $x=c$.

Exp) Find the equation of the tangent line to $f(x) = \frac{1}{x}$ at $x=3$.

Solution:

$$m_{tan} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$f(x) = \frac{1}{x}$

$f(x+\Delta x) = \frac{1}{x+\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x}{\scriptstyle x} - \frac{1}{x}{\scriptstyle (x+\Delta x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{\cancel{x} - \cancel{x} - \cancel{\Delta x}}{x(x+\Delta x)}}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{x(x+\Delta x)} = \frac{-1}{x^2}$$

Eval. $m_{tan}$ at $x=3$: $f'(3) = \frac{-1}{3^2} = \frac{-1}{9}$

Eq. of the tanget line at $P(3, \frac{1}{3})$ : $y - \frac{1}{3} = \frac{-1}{9}(x-3)$

Exp) Find the equation of the normal to $y = f(x) = \frac{1}{x}$ at $x = 3$.

$$m_{tan}\Big|_{x=3} = \frac{-1}{m_{normal}\Big|_{x=3}} \qquad m_{tan}\Big|_{x=3} = \frac{-1}{9}$$

$$m_{normal} = \frac{-1}{m_{tan}} = \frac{-1}{\frac{-1}{9}} = 9$$

$x = 3$

$f(x) \Rightarrow f(3) = \frac{1}{3}$

$P\left(3, \frac{1}{3}\right)$

$$y - \frac{1}{3} = 9(x - 3)$$

Exp) Differentiate $f(t) = 4 - t^2$

Solution: $f'(t) = ?$

$f(t) = 4 - t^2$

$f(t+\Delta t) = 4 - (t+\Delta t)^2 = 4 - t^2 - 2t\cdot\Delta t - \Delta t^2$

$$\lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

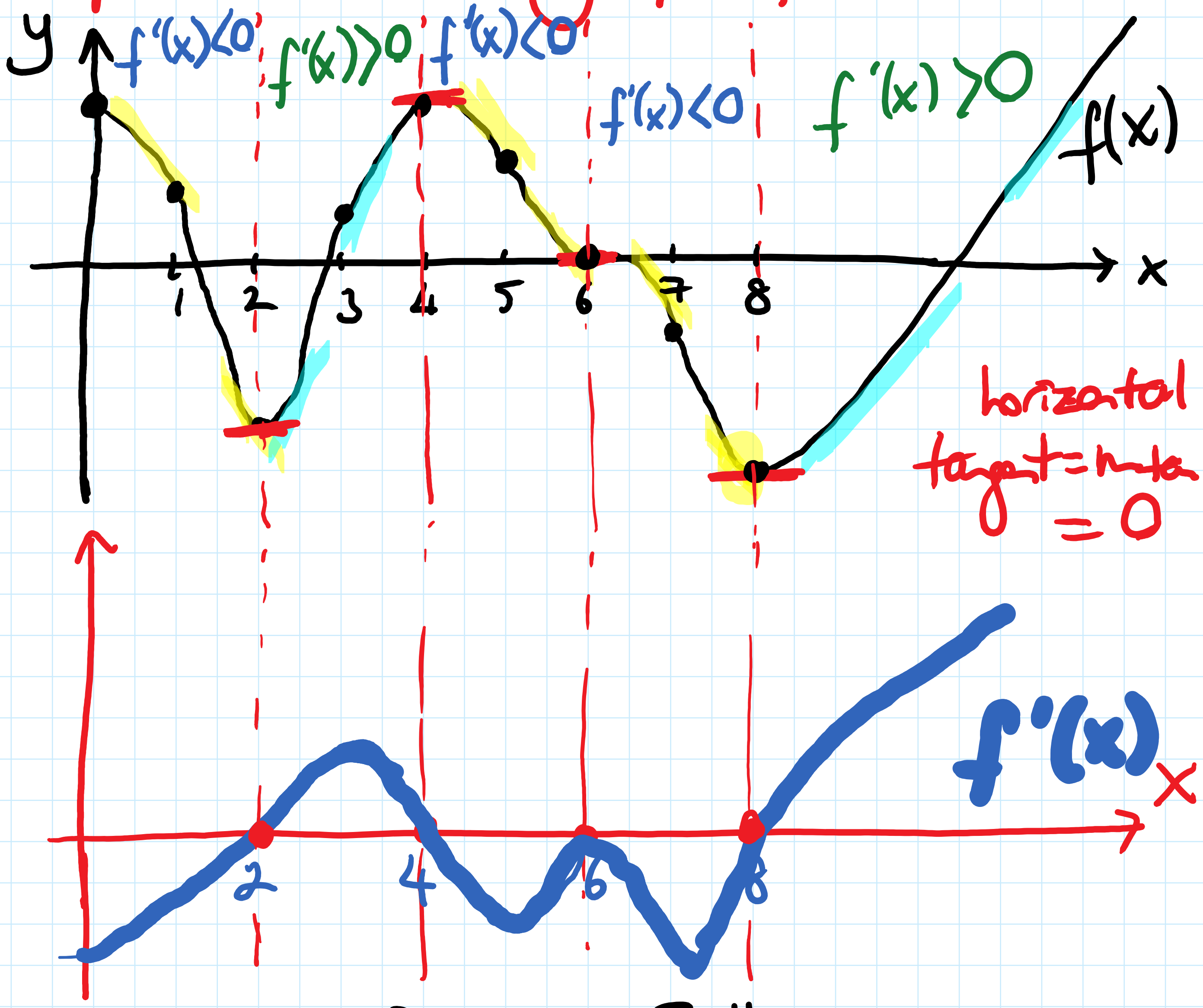
$$\lim_{\Delta t \to 0} \frac{4 - t^2 - 2t\cdot\Delta t - \Delta t^2 - (4 - t^2)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\cancel{4} \cancel{- t^2} - 2t\cdot\Delta t - \Delta t^2 \cancel{- 4} \cancel{+ t^2}}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{-2t\cdot\Delta t - \Delta t^2}{\Delta t} = \lim_{\Delta t \to 0} \frac{\cancel{\Delta t}(-2t - \Delta t)}{\cancel{\Delta t}}$$

$$\lim_{\Delta t \to 0} (-2t - \Delta t) = -2t$$

Exp) How do we graph $f'(x)$ from $f(x)$?




Steps to Follow:

1) Identify which interval: $f'(x)=0$

$f'(2)=f'(4)=f'(6)=f'(8)=0$

2) Identify which intervals $f'(x)>0$

3) Identify which intervals $f'(x)<0$

recall: negative slope \ ; positive slope /