3.2–3.3 Techniques of Differentiation

Goal: To compute derivatives indirectly by using formulas without evaluating any limits.

Table to memorize

Type of function	$f(x)$	$f'(x)$
constant	c	0
power	x^n	$n \cdot x^{n-1}$
exponential	e^x	e^x
Natural Log	$\ln x$	$\frac{1}{x}$
Trig. Functions	$\sin x$	$\cos x$
	$\cos x$	$-\sin x$
	$\tan x$	$\sec^2 x$
	$\cot x$	$-\csc^2 x$
	$\csc x$	$-\csc x \cdot \cot x$
	$\sec x$	$\sec x \cdot \tan x$

Rules for combining derivatives - Procedural Forms

$f(x)$	$f'(x)$	
$f \pm g$	$f' \pm g'$	Sum/Diff. Rules
$f \cdot g$	$f' \cdot g + f \cdot g'$	Product Rule
$\frac{f}{g}$	$\frac{f' \cdot g - f \cdot g'}{g^2}$	Quotient Rule "order matters"

" $f(x) = \log_b x$ $\qquad f'(x) = \frac{1}{x \cdot \ln b}$ "

Higher-Order Derivatives

# Derivative	Leibniz Notation	Lagrange Notation
1st Der.	$\frac{df}{dx}$	f' (f prime)
2nd Der.	$\frac{d^2f}{dx^2}$	f'' (f double prime)
3rd Der.	$\frac{d^3f}{dx^3}$	f''' (f triple prime)
4th Der.	$\frac{d^4f}{dx^4}$	$f^{(4)}$
⋮	⋮	⋮
Nth Der.	$\frac{d^Nf}{dx^N}$	$f^{(N)}$

$$f^4(x) = (f(x))^4 \quad \text{vs.} \quad f^{(4)}(x)$$

4th power vs. 4th der.

Notations for Derivatives

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$ (text)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ (MathXL)

$$m_{tan} = \lim_{h \to 0} (m_{sec})$$

Evaluate $f'(x)$ at $P(a, f(a))$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

OR

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Exp) Verify $\frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$

" quotient rule "

$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$

Solution: $\csc x = \frac{1}{\sin x}$

$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1 \to f}{\sin x \to g}\right)$

$= \frac{1' \cdot \sin x - 1 \cdot (\sin x)'}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = \frac{-\cos x}{\sin x \cdot \sin x}$

$= \frac{-\cot x}{\sin x} = -\cot x \cdot \frac{1}{\sin x} = -\cot x \cdot \csc x$

Exp) Calculate $\frac{d}{dx}\left(\frac{1}{2\sqrt{x}} + \frac{x^2}{4} + \pi\right)$

$\frac{1}{2\sqrt{x}} = \frac{1}{2} \cdot x^{-1/2}$

$= \frac{d}{dx}\left(\frac{1}{2} \cdot x^{-1/2}\right) + \frac{d}{dx}\left(\frac{x^2}{4}\right) + \frac{d}{dx}(\pi)$

strategy
re-write radicals as power

$= \frac{1}{2} \cdot \frac{-1}{2} \cdot x^{-\frac{1}{2}-1} + \frac{1}{4} \cdot 2 \cdot x^1 + 0$

$(x^n)' = n \cdot x^{n-1}$

$= \frac{-1}{4} \cdot x^{-3/2} + \frac{x}{2}$

Exp) Calculate $\frac{d}{dx}\left(\frac{2x^5-3x^2+11}{x^3}\right)$

Solution:

$$\frac{d}{dx}\left(\frac{2x^5}{x^3}\right)-\frac{d}{dx}\left(\frac{3x^2}{x^3}\right)+\frac{d}{dx}\left(\frac{11}{x^3}\right)$$

$$=\frac{d}{dx}(2x^2)-\frac{d}{dx}\left(\frac{3}{x}\right)+\frac{d}{dx}\left(\frac{11}{x^3}\right)$$

$$=2\cdot 2\cdot x^1-\frac{d}{dx}(3\cdot x^{-1})+\frac{d}{dx}(11\cdot x^{-3})$$

$$=4x-3\cdot(-1)\cdot x^{-2}+11\cdot(-3)\cdot x^{-4}$$

$$=4x+3x^{-2}-33x^{-4}$$

$$=4x+\frac{3}{x^2}-\frac{33}{x^4}$$

Exp) $g(x) = \frac{4x-7}{3-x^2}$ $g'(x) = ?$

Solution: use the quotient rule:

$$g(x) = \frac{f(x)}{g(x)} \; ; \; g'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$$

$$g'(x) = \frac{(4x-7)'(3-x^2) - (4x-7) \cdot (3-x^2)'}{(3-x^2)^2}$$

$$= \frac{4(3-x^2) + (4x-7) \cdot (+2 \cdot x^1)}{(3-x^2)^2}$$

$$= \frac{12 - 4x^2 + 8x^2 - 14x}{(3-x^2)^2} = \frac{4x^2 - 14x + 12}{(3-x^2)^2}$$

Exp) Calculate $F'(x)$ for $F(x) = \dfrac{x\sqrt{x}\cdot \tan x}{e^x - e^3}$

Solution: re-write $x\sqrt{x}$ as $x \cdot x^{\frac{1}{2}} = x^{3/2}$

$$F(x) = \frac{x^{3/2}\cdot \tan x}{e^x - e^3} \quad \begin{matrix} \to f \\ \to g \end{matrix} \qquad F'(x) = \frac{f' \cdot g - f \cdot g'}{g^2}$$

$$= \frac{\boxed{(x^{3/2}\cdot \tan x)'}\cdot (e^x - e^3) - (x^{3/2}\cdot \tan x)\cdot (e^x - e^3)'}{(e^x - e^3)^2}$$

$$= \frac{\left((x^{3/2})' \cdot \tan x + x^{3/2} (\tan x)'\right)(e^x - e^3) - (x^{3/2}\cdot \tan x)(e^x)}{(e^x - e^3)^2}$$

$$= \frac{\left(\frac{3}{2} \cdot x^{\frac{1}{2}} \cdot \tan x + x^{3/2} \sec^2 x\right)(e^x - e^3) - (x^{3/2}\cdot \tan x)\cdot e^x}{(e^x - e^3)^2}$$

Exp) Find $f'''(x)$ if $f(x)=\pi \cdot e^x+\cos x-2\sqrt{x}$

Solution: $f(x)=\pi \cdot e^x+\cos x-2 \cdot x^{\frac{1}{2}}$

$f'(x)=\pi \cdot e^x-\sin x-2 \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}}$

$f'(x)=\pi \cdot e^x-\sin x-x^{-\frac{1}{2}}$

$f''(x)=\pi \cdot e^x-\cos x-\left(-\frac{1}{2}\right) \cdot x^{-3/2}$

$f''(x)=\pi \cdot e^x-\cos x+\frac{1}{2} \cdot x^{-3/2}$

$f'''(x)=\pi \cdot e^x+\sin x+\frac{1}{2}\left(\frac{-3}{2}\right) \cdot x^{-5/2}$

$f'''(x)=\pi \cdot e^x+\sin x-\frac{3}{4} \cdot x^{-5/2}$

$\sqrt{x}=x^{1/2}$

$\sqrt[3]{x}=x^{\frac{1}{3}}$

$\sqrt[3]{x^2}=x^{\frac{2}{3}}$

Exp) Pg 152 Q31

Find the equation for the tangent line to $y = f(x) = \frac{x^2+5}{x+5}$ at $x=1$.

Solution: Steps

$m_{tan} = f'(x)$

$m_{tan}\Big|_{x=1} = f'(1)$

Use $x_1 = 1$ to find $f(1) = f(x_1)$ to $f(x)$

Set up eq. for the tangent line at $x=1$

$y - y_1 = m_{tan}(x - x_1)$

$$f'(x) = \frac{2x(x+5) - (x^2+5)\cdot 1}{(x+5)^2}$$

m_{tan} at $x=1$

$$f'(1) = \frac{2\cdot 1(1+5) - (1^2+5)\cdot 1}{(1+5)^2} = \frac{2\cdot 6 - 6}{6^2} = \frac{6}{36} = \frac{1}{6}$$

$f(x) = \frac{x^2+5}{x+5}$ $f(1) = \frac{1+5}{1+5} = 1$ $P(1,1)$

$m_{tan}\Big|_{x=1} = \frac{1}{6}$

eq. of tangent line to f(x) at (1,1)

$$\boxed{y-1 = \frac{1}{6}(x-1).}$$

VS.

eq. of normal line to f(x) at (1,1)

$$y-1 = -6(x-1)$$

Exp) Calculate $h'(x)$ for $h(x) = \cos x \cdot \ln x$

Solution:

$$h'(x) = ((\cos x)' \cdot \ln x + (\cos x) \cdot (\ln x)'$$

$$= -\sin x \cdot \ln x + \cos x \cdot \frac{1}{x}$$

$\cos\left(\frac{x}{x}\right) = \cos 1$

~~Cos 1~~

~~Cos~~

don't simplify

Exp) Find equation for the tangent line to the curve with eq. $y=x^4-2x+1$ that's parallel to the line $2x-y-3=0$

Solution: re-write $2x-y-3=0$ as

$y=2x-3$

$m=2$

$m_{tan}=2$ //

$f'(x)=2 \Big|_{x=?}$

$f'(x)=4x^3-2$

$f'(x)=2$

$4x^3-2=2 \Rightarrow 4x^3-4=0$

$4(\underbrace{x^3-1}_{0})=0$

$x^3-1=0 \Rightarrow \boxed{x=1}$

$f(x)=x^4-2x+1$

$f(1)=1^4-2\cdot1+1=0$

P(1,0)

$y-0=2(x-1)$

Exp) Find normal line to $f(x) = x^2 \cdot \tan x$ at $x = \frac{\pi}{4}$.

Solution: find $m_{tan}\Big|_{x=\frac{\pi}{4}}$

$\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = \frac{1}{\frac{\sqrt{2}}{2}}$

$m_{tan} = f'(x)$

$f'(x) = 2x \cdot \tan x + x^2 \cdot \sec^2 x$

$$f'\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\pi}{4} \cdot \tan\left(\frac{\pi}{4}\right) + \left(\frac{\pi}{4}\right)^2 \cdot \sec^2\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2} \cdot 1 + \frac{\pi^2}{16} \cdot \left(\frac{2}{\sqrt{2}}\right)^2$$

$$= \frac{\pi}{2} + \frac{\pi^2}{16} \cdot \frac{4}{2}^2 = \frac{\pi}{\underset{(4)}{2}} + \frac{\pi^2}{8}$$

$$m_{tan}\Big|_{x=\frac{\pi}{4}} = \frac{4\pi + \pi^2}{8}$$

$$m_{normal} = \frac{-1}{m_{tan}} = \frac{-1}{\frac{4\pi+\pi^2}{8}} = \frac{-8}{4\pi+\pi^2}$$

$$m_{normal} = \frac{-8}{4\pi+\pi^2}$$

$f(x) = x^2 \cdot \tan x$ $\quad x = \frac{\pi}{4}$ $\quad y = f\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16} \cdot \tan\left(\frac{\pi}{4}\right)$

$$f\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16}$$

The equation of the normal line to $f(x)$ at $x = \frac{\pi}{4}$:

$$y - \frac{\pi^2}{16} = \frac{-8}{4\pi+\pi^2}\left(x - \frac{\pi}{4}\right)$$

Exp) a) Calculate $f'(x)$ for $f(x) = x^2 \cdot e^x$

Solution:

$$f'(x) = 2x \cdot e^x + x^2 \cdot e^x = e^x \cdot x(2+x)$$

Part b) For what values of x does $f'(x) = 0$

$$f'(x) = 0 = e^x \cdot x(2+x)$$

$e^x = 0 \Rightarrow$ NEVER! $e^x \neq 0$

$x = 0 \Rightarrow x = 0$

$2 + x = 0 \Rightarrow x = -2$

Exp) Find the equation for the tangent line to $f(x) = e^x \cdot \cos x$ at $x = 0$.

Solution:

$$f'(x) = e^x \cdot \cos x + e^x \cdot (-\sin x)$$

$$f'(0) = e^0 \cdot \cos 0 + e^0 \cdot (-\sin 0)$$

$$= 1 \cdot 1 + 1 \cdot 0 = 1 = m_{tan} \text{ at } x=0$$

when $x = 0$ $f(0) = e^0 \cdot \cos 0 = 1 \cdot 1 = 1$

$$y - 1 = 1(x - 0)$$