3.5. Chain Rule

In 3.2–3.3; we calculated the derivatives of $\ln x$, $\sin x$, x^2-3.

How do we differentiate functions:

$\ln(x-2)$, $\sin(\cos x)$, $(x^2-3)^2$

Theorem: If f and g are differentiable then the derivative $f(g(x))$ is:

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

Exp) Differentiate $y=(\underbrace{3x^4-7x+5}_{g(x)})^3$; $f(x)$

$y = (g(x))^3$

$$\boxed{f(x) = (g(x))^3}$$

$g(x) = 3x^4-7x+5$

$f'(g(x)) \cdot g'(x) =$

$3 \cdot g(x)^2 \cdot (3x^4-7x+5)'$

$= 3(3x^4-7x+5)^2 \cdot (3 \cdot 4 \cdot x^3 - 7)$

Exp) $y = \sin(x^2)$

Solution: $f(x) = \sin(g(x))$

$g(x) = x^2$

$$f'(g(x)) \cdot g'(x) = (\sin(g(x)))' \cdot (x^2)'$$

$$= \cos(g(x)) \cdot 2x$$

$$= \cos(x^2) \cdot \underbrace{2x}$$

$$= 2x \cdot \cos(x^2)$$

Exp) Differentiate

$$h(x) = \sqrt[4]{\frac{x}{1-3x}} = \left(\frac{x}{1-3x}\right)^{\frac{1}{4}}$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

Solution:

$$f(x) = [g(x)]^{\frac{1}{4}}$$

$$g(x) = \frac{x}{1-3x}$$

$$f'(g(x)) \cdot g'(x) = \frac{1}{4}[g(x)]^{\frac{-3}{4}} \cdot \left(\frac{x}{1-3x}\right)'$$

$$= \frac{1}{4}\left[\frac{x}{1-3x}\right]^{-3/4} \cdot \left(\frac{1 \cdot (1-3x) - x(-3)}{(1-3x)^2}\right)$$

$$= \frac{1}{4} \frac{(x)^{-3/4}}{(1-3x)^{-3/4}} \cdot \left(\frac{1 \cancel{-3x} \cancel{+3x}}{(1-3x)^2}\right)$$

$$x^m \cdot x^n = x^{m+n}$$

$$= \frac{1}{4} \cdot \frac{x^{-3/4}}{(1-3x)^{-3/4+2}}$$

$$= \frac{x^{-3/4}}{4(1-3x)^{5/4}}$$

$$\frac{-3}{4} + \frac{2}{1(4)}$$

$$\frac{-3}{4} + \frac{8}{4} = \frac{5}{4}$$

$$= \frac{1}{4 \cdot x^{3/4} \cdot (1-3x)^{5/4}}$$

Exp) Differentiate

$$g(x) = \underbrace{\cos(x^2)}_{} + \underbrace{5\left(\frac{3}{x}+4\right)^6}_{}$$

$\cos(x^2) \rightarrow g(x) = x^2, \quad f(x) = \cos(g(x))$

$$f'(g(x)) \cdot g'(x) = (\cos(g(x)))' \cdot (x^2)'$$

$$= -\sin(g(x)) \cdot 2x$$

$$\boxed{= -2x \cdot \sin(x^2)}$$

$5\left(\frac{3}{x}+4\right)^6 \rightarrow g(x) = \frac{3}{x}+4, \quad f(x) = 5(g(x))^6$

$$f'(g(x)) \cdot g'(x) = 5 \cdot 6 \cdot g(x)^5 \cdot \left(\frac{3}{x}+4\right)'$$

$$= 30 \cdot \left(\frac{3}{x}+4\right)^5 \cdot \left(3 \cdot x^{-1}+4\right)'$$

$$= 30\left(\frac{3}{x}+4\right)^5 \cdot \left(3 \cdot (-1) \cdot x^{-2}\right)$$

$$\boxed{= -90\left(\frac{3}{x}+4\right)^5 \cdot x^{-2}}$$

$$-2x \cdot \sin(x^2) - 90\left(\frac{3}{x}+4\right)^5 \cdot x^{-2}$$

Exp) Differentiate $p(x)=\frac{\tan 7x}{(1-4x)^5}$

use quotient rule: $\left(\frac{f}{g}\right)'=\frac{f'\cdot g-f\cdot g'}{g^2}$

$$p'(x)=\frac{(\tan 7x)'\cdot(1-4x)^5-(\tan 7x)\cdot\left[(1-4x)^5\right]'}{\left[(1-4x)^5\right]^2}$$

$(x^m)^n=x^{m\cdot n}$

$$=\frac{\sec^2(7x)\cdot 7(1-4x)^5-\tan 7x\left[5(1-4x)^4\cdot(-4)\right]}{(1-4x)^{10}}$$

$$=\frac{7\cdot\sec^2(7x)\cdot(1-4x)^5+20\cdot\tan(7x)\cdot(1-4x)^4}{(1-4x)^{10}}$$

$$=\frac{\cancel{(1-4x)^4}\left[7\cdot\sec^2(7x)\cdot(1-4x)+20\cdot\tan(7x)\right]}{\cancel{(1-4x)^{10}}\ (1-4x)^6}$$

$$=\frac{7\cdot\sec^2(7x)}{(1-4x)^5}+\frac{20\cdot\tan(7x)}{(1-4x)^6}$$

Exp) Differentiate $y = e^{-3x} \cdot \sin x$

$(f \cdot g)' = f' \cdot g + f \cdot g'$ (Product Rule)

$$y' = (e^{-3x})' \cdot \sin x + \underline{(e^{-3x}) \cdot (\sin x)'}$$

$$= e^{-3x} \cdot (-3x)' \cdot \sin x + e^{-3x} \cdot \cos x$$

$$\left[e^{g(x)}\right]' = \left(e^{g(x)}\right)' \cdot (g(x))'$$

$$\left[e^{x}\right]' = e^{x} \cdot 1$$

$$= -3 \cdot e^{-3x} \cdot \sin x + e^{-3x} \cdot \cos x$$

$$= e^{-3x}(-3\sin x + \cos x)$$

Exp) Differentiate $y = \ln(x^3 + x)$

$(\ln x)' = \dfrac{1}{x}$; $(\ln(g(x)))' = \dfrac{g'(x)}{g(x)}$

$(\ln x)' = \dfrac{x'}{x} = \dfrac{1}{x}$

$y = \ln(g(x))$

$g(x) = x^3 + x$

$$y' = \frac{(x^3+x)'}{x^3+x} = \boxed{\frac{3x^2+1}{x^3+x}}$$

Exp) Differentiate $y = \sin(e^x)\cdot \cos x$

Product rule: $(f\cdot g)' = f'g + fg'$

$$y' = \underbrace{(\sin(e^x))'}\cdot \cos x + (\sin(e^x))\cdot(\cos x)'$$

$$= \underbrace{\cos(e^x)}\cdot e^x\cdot \cos x + \underbrace{\sin(e^x)}\cdot(-\sin x)$$

Exp) Differentiate $y = \sqrt{\dfrac{x^3}{1-x}} = \left(\dfrac{x^3}{1-x}\right)^{\frac{1}{2}}$

Power rule, quotient rule, chain rule

$\sqrt[3]{x} = x^{\frac{1}{3}}$

$\sqrt[2]{x} = x^{\frac{1}{2}}$

$$y' = \underbrace{\frac{1}{2}\left(\frac{x^3}{1-x}\right)^{-\frac{1}{2}}}_{\text{outside}}\cdot \underbrace{\left[\frac{x^3}{1-x}\right]'}_{\text{inside}}$$

$$= \frac{1}{2}\left(\frac{x^3}{1-x}\right)^{-\frac{1}{2}}\cdot\left[\frac{3x^2(1-x) - x^3(1-x)'}{(1-x)^2}\right]$$

$$= \frac{1}{2}\left(\frac{x^3}{1-x}\right)^{-\frac{1}{2}}\cdot\left[\frac{3x^2-3x^3-x^3\cdot(-1)}{(1-x)^2}\right]$$

$$= \frac{1}{2}\,\frac{x^{-3/2}}{(1-x)^{-\frac{1}{2}}}\cdot\left[\frac{3x^2-2x^3}{(1-x)^2}\right]$$

$$\frac{1}{2}\,\frac{x^{-3/2}}{(1-x)^{-\frac{1}{2}}}\cdot\left[\frac{3x^2-2x^3}{(1-x)^2}\right]$$

$$=\frac{1}{2}\cdot\frac{x^{-3/2}\cdot x^2(3-2x)}{(1-x)^{-\frac{1}{2}+2}}$$

$x^{-\frac{3}{2}+2}=x^{\frac{1}{2}}$

$(1-x)^{3/2}$

$$=\frac{1}{2}\cdot\frac{x^{\frac{1}{2}}\,(3-2x)}{(1-x)^{3/2}}$$

Exp) Find the x-coordinate of each point on the graph of $f(x)=x^2\cdot(4x+5)^3$ where the tangent line is horizontal.

Solution:

$f'(x)=0$ solve it for x.

$$f'(x) = (x^2)'(4x+5)^3 + x^2\cdot\left[(4x+5)^3\right]'_{(4x+5)'}$$

$$= 2x(4x+5)^3 + x^2\cdot 3(4x+5)^2\cdot \boxed{4}$$

$$= 2x(4x+5)^2\left[(4x+5) + x\cdot 3\cdot 2\right]$$

$$= 2x(4x+5)^2(10x+5)$$

$$f'(x)=0 = \underbrace{2x}_{0}\cdot\underbrace{(4x+5)^2}_{0}\cdot\underbrace{(10x+5)}_{0}$$

$2x=0 \Rightarrow x=0$

$(4x+5)^2=0 \Rightarrow 4x+5=0 \Rightarrow 4x=-5 \Rightarrow x=\frac{-5}{4}$

$(10x+5)=0 \Rightarrow 10x=-5 \Rightarrow x=\frac{-5}{10}=\frac{-1}{2}$

(x, y)

Exp) Find the x-coordinate of each point on the graph of $V(x) = \frac{\ln\sqrt{x}}{x^2}$ where the tangent line is horizontal.

Solution: horizontal tangent means

$V'(x) = 0$

$$V'(x) = \frac{(\ln\sqrt{x})' \cdot (x^2) - (\ln\sqrt{x}) \cdot (x^2)'}{(x^2)^2}$$

$$(\ln\sqrt{x})' = \frac{(\sqrt{x})'}{\sqrt{x}} = \frac{(x^{\frac{1}{2}})'}{\sqrt{x}} = \frac{\frac{1}{2} \cdot x^{-\frac{1}{2}}}{x^{\frac{1}{2}}} = \frac{1}{2} x^{-1}$$

$$= \frac{\frac{1}{2} \cdot x^{-1} \cdot (x^2) - \ln(\sqrt{x}) \cdot 2x}{x^4}$$

$$= \frac{\frac{1}{2} \cdot x - \ln\sqrt{x} \cdot 2x}{x^4}$$

$$= \frac{\cancel{x}\left(\frac{1}{2} - 2 \cdot \ln\sqrt{x}\right)}{\cancel{x^4}\ x^3}$$

$$V'(x) = 0 = \frac{\frac{1}{2} - 2\cdot\ln\sqrt{x}}{x^3}$$

cross multiply:

$$0\cdot x^3 = \frac{1}{2} - 2\cdot\ln\sqrt{x}$$

$$0 = \frac{1}{2} - 2\cdot\ln x^{\frac{1}{2}}$$

$$0 = \frac{1}{2} - 2\cdot\frac{1}{2}\cdot\ln x$$

$$0 = \frac{1}{2} - \ln x$$

$$+\frac{1}{2} = +\ln x$$

$$\frac{1}{2} = \ln x$$

$$e^{\frac{1}{2}} = x$$

ln/log property:

$$\ln\sqrt{x} = \ln(x^{\frac{1}{2}})$$

$$= \frac{1}{2}\cdot\ln x$$

$$\ln_e x = \frac{1}{2} \Rightarrow x = e^{\frac{1}{2}}$$

Exp) B type Q from text

Differentiate $g(t) = t^2 \cdot e^{-t} + (\ln t)^2$

$$g'(t) = 2t \cdot e^{-t} + t^2 (e^{-t})' + 2 \cdot \ln t \cdot (\ln t)'$$

$$= 2t \cdot e^{-t} + t^2 (e^{-t}) \cdot (-t)' + 2 \cdot \ln t \cdot \frac{1}{t}$$

$$\left[e^{f(x)}\right]' = e^{f(x)} \cdot f'(x) \qquad (\ln t)' = \frac{t'}{t} = \frac{1}{t}$$

$$= 2t \cdot e^{-t} + t^2 (e^{-t}) \cdot (-1) + \frac{2}{t} \cdot \ln t$$

$$= t \cdot e^{-t}(2-t) + \frac{2}{t} \cdot \ln t$$

$\cancel{\frac{\ln t}{t}}$ $\cancel{\ln 1}$

be careful!