3.6. Implicit Differentiation

Goal: To differentiate functions that can be implicitly defined when derivative exist.

$f(x) = y = \sqrt{1-x^2}$ y is explicitly defined as a function of x for $-1 \le x \le 1$

the same function can also be implicitly defined as: $y^2 = (\sqrt{1-x^2})^2$

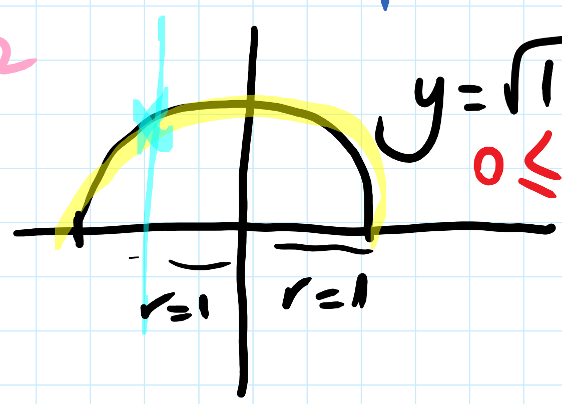
$y^2 = 1 - x^2$

y is still a function of x. $\boxed{x^2+y^2=1}$ (eq. of a circle which has a center on the origin, radius of 1)

$\boxed{x^2+y^2=r^2}$

fails the vertical line test.

$x^2+y^2=1^2 \Longrightarrow$



$y=\sqrt{1-x^2}$

$0 \le y \le 1$

Procedure for Implicit Differentiation

Suppose an equation defines y implicitly as a differentiable function of x.

To find $\frac{dy}{dx}$ (y')

Step 1) Differentiate both sides of the equation w/ respect to x. Remember y is really a function of x for part of the curve and use the chain rule when differentiating terms containing y.

Step 2) Solve the differentiated equation algebraically for $\frac{dy}{dx}$ (y')

Exp) $y=\sqrt{1-x^2}$ explicit f of x

$x^2+y^2=1$ implicitly defined f of x

$$y=\sqrt{1-x^2}$$

$$y'=\left(\sqrt{1-x^2}\right)'$$

$$=\left[(1-x^2)^{1/2}\right]'$$

$$=\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\cdot(-2x)$$

$$=\boxed{\frac{-x}{\sqrt{1-x^2}}}$$

$$(x^2+y^2)'=(1)'$$

$$\cancel{2x}+2y\cdot y'=0$$

$$\cancel{-2x} \qquad -2x$$

$$\frac{\cancel{2y}\cdot y'}{\cancel{2y}}=\frac{-\cancel{2}x}{\cancel{2}y}$$

$$y'=\frac{-x}{y}$$

$$y'=\boxed{\frac{-x}{\sqrt{1-x^2}}}$$

We should obtain the same y'.

Exp) Find y' for $\underbrace{x^2 \cdot y^3 - 6 = 5y^3 + x}$

is the same as: $y = \left(\dfrac{x+6}{x^2-5}\right)^{1/3}$ power rule quotient rule chain rule

Solution:

Step 1:

$$x^2 \cdot y^3 - 6 = 5y^3 + x$$

$$(x^2 \cdot y^3)' - 6' = (5y^3)' + (x)'$$

Diff. w/ respect to x.

$$2x \cdot y^3 + x^2 \cdot (y^3)' = 15y^2 \cdot y' + 1$$

$$2x \cdot y^3 + x^2 \cdot 3y^2 \cdot y' = 15y^2 \cdot y' + 1$$

Step 2:

$$3x^2y^2 \cdot y' - 15y^2 \cdot y' = 1 - 2x \cdot y^3$$

$$y' \cdot [3x^2y^2 - 15y^2] = 1 - 2xy^3$$

$$y' = \frac{1-2xy^3}{3x^2y^2 - 15y^2} = \frac{2xy^3 - 1}{15y^2 - 3x^2y^2}$$

Exp) Find y' for $x^2y+2y^3=3x+2y$

$$(x^2\cdot y)'+(2y^3)'=(3x)'+(2y)'$$

find the derivative w/ respect to x

$$2x\cdot y+x^2\cdot y'+6y^2\cdot y'=3+2\cdot y'$$

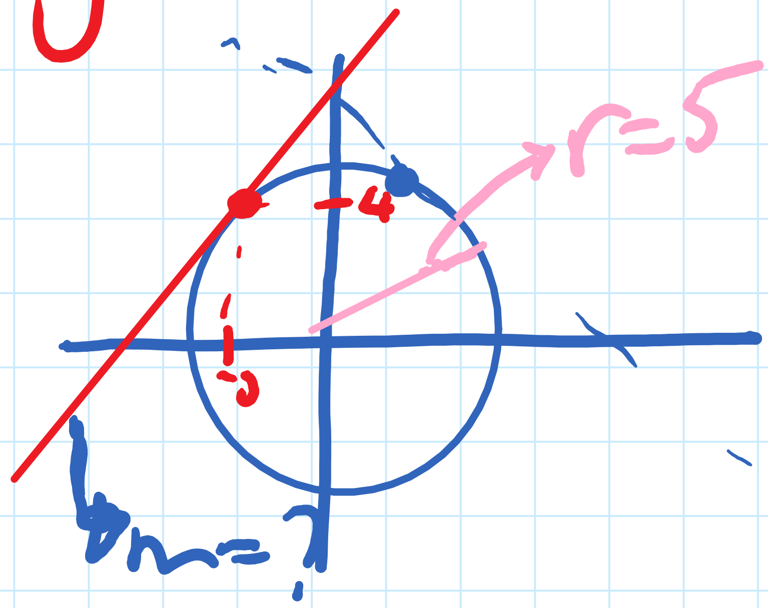
$$x^2\cdot y'+6y^2\cdot y'-2\cdot y'=3-2xy$$

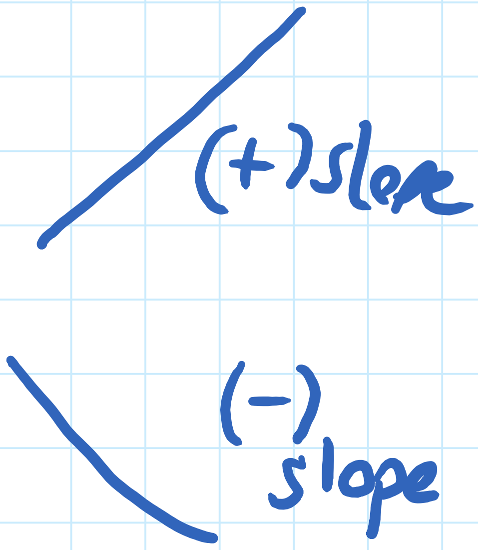
$$\frac{y'(x^2+6y^2-2)}{x^2+6y^2-2}=\frac{3-2xy}{x^2+6y^2-2}$$

$$y'=\frac{3-2xy}{x^2+6y^2-2}$$

Exp) Find the slope of the tangent line to the curve $x^2+y^2=25$ at $P(-3,4)$

Solution:





$$(x^2+y^2)'=(25)'$$

$$2x+2y\cdot y'=0$$

$$\begin{array}{rr} -2x & -2x \end{array}$$

$$\frac{2y\,y'}{2y}=\frac{-2x}{2y}$$

$$\boxed{y'=\frac{-x}{y}}$$

$$y'\Big|_{(-3,4)}=\frac{-(-3)}{4}=\frac{3}{4}$$

Exp) What if $x^2+y^2=-1$, $y'=?$

If we follow the procedure blindly we obtain $y'=\frac{-x}{y}$; however, in the world of real number, there're values of x & y that makes $x^2+y^2=-1$ a valid statement. Derivative DNE.

Exp) Find y' for : $\sin(x^2+y) = y^2 \cdot (3x+1)$

Solution: $\left(\sin(x^2+y)\right)' = \left(y^2 \cdot (3x+1)\right)'$

$$\cos(x^2+y) \cdot (2x+y') = 2y \cdot y' \cdot (3x+1) + y^2 \cdot 3$$

$$2x \cdot \cos(x^2+y) + \cos(x^2+y) \cdot y' = 2y \cdot y' \cdot 3x + 2y \cdot y' + 3y^2$$

$$2x \cdot \cos(x^2+y) - 3y^2 = 6xy \cdot y' + 2y \cdot y' - \cos(x^2+y) \cdot y'$$

$$2x \cdot \cos(x^2+y) - 3y^2 = y' \cdot \left(6xy + 2y - \cos(x^2+y)\right)$$

$$\boxed{y' = \frac{2x \cdot \cos(x^2+y) - 3y^2}{6xy + 2y - \cos(x^2+y)}}$$

Expl Find y'' when $x^2+y^2=10$

Solution: $(x^2+y^2)' = 10'$

$$\cancel{2x} + 2y\cdot y' = 0$$
$$\underline{-\cancel{2x} \qquad\qquad -2x}$$

$$\frac{2y\cdot y'}{2y} = \frac{-2x}{2y} \Rightarrow \boxed{y' = \frac{-x}{y}}$$

$$y'' = (y')' \Rightarrow y'' = \left(\frac{-x}{y}\right)'$$

$$y'' = \frac{-1\cdot y - (-x)\cdot y'}{y^2}$$

$$y'' = \frac{-y + x\cdot y'}{y^2}$$

Given $x^2+y^2=10$

$$y'' = \frac{-y + x\cdot\left(\frac{-x}{y}\right)}{y^2} = \frac{-y_{\frac{}{1(y)}} - \frac{x^2}{y}}{y^2}$$

$$= \frac{\frac{-y^2-x^2}{y}}{y^2} = \frac{-x^2-y^2}{y^3}$$

$$= -\frac{(x^2+y^2)}{y^3} = \boxed{\frac{-10}{y^3} = y''}$$

Exp) Show that $(x^n)' = n \cdot x^{n-1}$, $x > 0$

Solution: $y = x^n$

$\ln y = \ln x^n$ $\qquad [\ln x^n = n \cdot \ln x]$

$(\ln y)' = (\ln x^n)'$

$(\ln y)' = (n \cdot \ln x)'$

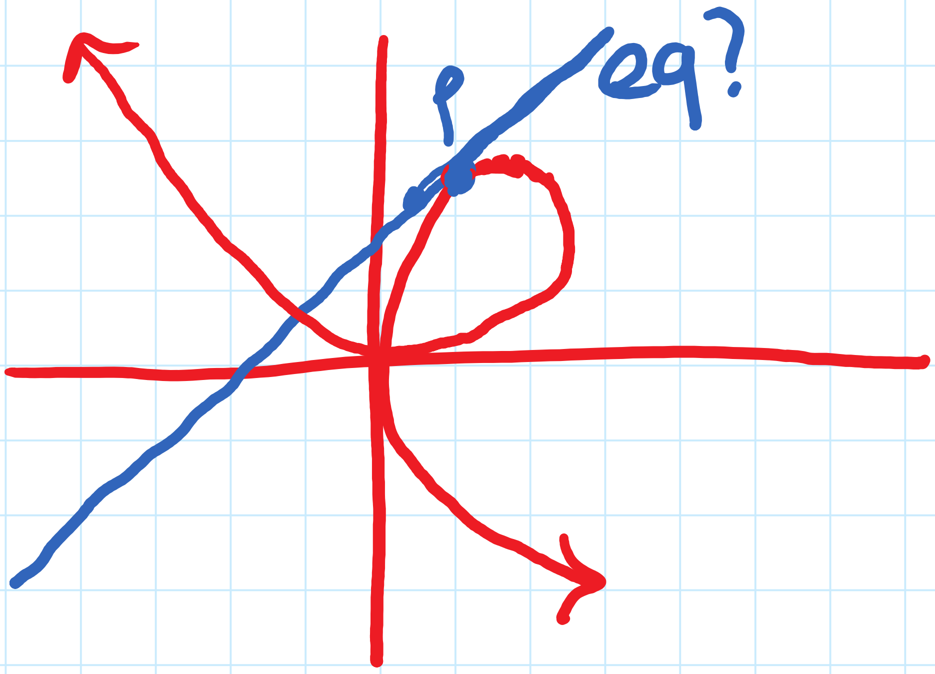
$y \cdot \frac{y'}{y} = n \cdot \frac{1}{x} \cdot y$

$y' = n \cdot \frac{1}{x} \cdot y$

$y' = n \cdot \frac{1}{x} \cdot x^n$

$\boxed{y' = n \cdot x^{n-1}}$

Exp) Find an equation of the tangent line to $x^3+y^3=3xy$ at $P\left(\frac{2}{3},\frac{4}{3}\right)$



$$(x^3+y^3)' = (3xy)'$$

$$3x^2+3y^2\cdot y' = 3\cdot y+3x\cdot y'$$

$$3x^2-3y = 3xy'-3y^2\cdot y'$$

$$\cancel{3}(x^2-y) = \cancel{3}y'(x-y^2)$$

$$y' = \frac{x^2-y}{x-y^2}$$

$$y'\Big|_{\left(\frac{2}{3},\frac{4}{3}\right)} = \frac{\left(\frac{2}{3}\right)^2-\frac{4}{3}}{\frac{2}{3}-\left(\frac{4}{3}\right)^2} = \frac{\frac{4}{9}-\frac{4}{3}_{(3)}}{\frac{2}{3}_{(3)}-\frac{16}{9}}$$

$$y'\Big|_{\left(\frac{2}{3},\frac{4}{3}\right)} = \frac{\frac{4}{9}-\frac{12}{9}}{\frac{6}{9}-\frac{16}{9}} = \frac{\frac{4-12}{9}}{\frac{6-16}{9}} = \frac{-8}{-10} = \frac{8}{10}$$

$$y-\frac{4}{3} = \frac{8}{10}\left(x-\frac{2}{3}\right)$$

The eq. of the tangent line to the curve at point P.

Exp) find y' for $f(x)=x^x$

Solution: $y = x^x$

(Use logarithmic differentiation)

$$\ln y = \ln x^x$$

$$\ln y = x \cdot \ln x$$

Then $y' \Rightarrow$ $(\ln y)' = (x \cdot \ln x)'$

$$y \cdot \frac{y'}{y} = \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) \cdot y$$

$$y' = (\ln x + 1) \cdot y$$

$$\boxed{y' = (\ln x + 1) \cdot x^x}$$

[Given $y = x^x$]

Exp) $\boxed{y=(1+\sin(2x))^{x^2}}$ find $\frac{dy}{dx}$ (y')

Solution:

Use logarithmic differentiation

$$\ln y = \ln(1+\sin(2x))^{x^2}$$

$$\ln y = x^2 \cdot \ln(1+\sin(2x))$$

$$\frac{dy/dx}{y} = 2x\cdot\ln(1+\sin(2x)) + x^2\cdot\frac{(1+\sin(2x))'}{1+\sin(2x)}$$

$$y\cdot\frac{dy/dx}{y} = \left(2x\cdot\ln(1+\sin(2x)) + x^2\cdot\frac{2\cdot\cos 2x}{1+\sin 2x}\right)\cdot y$$

$$\boxed{\frac{dy}{dx} = \left(2x\cdot\ln(1+\sin(2x)) + \frac{2x^2\cdot\cos 2x}{1+\sin 2x}\right)\cdot(1+\sin(2x))^{x^2}}$$

Exp) find y' for $\sin(x+y) = x+\cos y$

$$(\sin(x+y))' = (x+\cos y)'$$

$$\cos(x+y)\cdot(x+y)' = 1-\sin y\cdot y'$$

$$\cos(x+y)\cdot(1+y') = 1-\sin y\cdot y'$$

$$\cos(x+y)+\cos(x+y)\cdot y' = 1-\sin y\cdot y'$$

$$\cos(x+y)\cdot y'+\sin y\cdot y' = 1-\cos(x+y)$$

$$y'\cdot(\cos(x+y)+\sin y) = 1-\cos(x+y)$$

$$y' = \frac{1-\cos(x+)}{\cos(x+y)+\sin y}$$