3.8. Linear Approximation and Differentials

Tangent Line approximation (Linearization)

If $f(x)$ is differentiable at $x=a$;
The the tangent line at $P(a, f(a))$ on the graph of $y=f(x)$ has a slope of $m=f'(a)$ and an equation of $f'(a)=\dfrac{y-f(a)}{x-a}$;

OR $\quad y=f(a)+f'(a)(x-a)$

When x_1 is near a, the $f(x_1)$ must be close to the point on the tangent line to $y=f(x)$, at $x=x_1$.

$$f(x_1) \approx f(a)+f'(a)(x_1-a)$$

Linear approximation of f at $x=a$;

$$L(x)=f(a)+f'(a)(x-a).$$

The function $L(x)$ is a linearization of $f(x)$ at $x=a$.

Exp) Find the equation of the tangent line to $f(x)=y=x^3$ at $x=2$.

$f(2)=2^3=8$

$(2,8)$

$f'(x)=3x^2$

$f'(2)=3\cdot 2^2=12$

$y-y_1=m(x-x_1)$

$y-8=12(x-2)$

Exp) Find linear approximation (linearization) to $y=x^3$ at $x=2$.

$$\boxed{L(x)=f(a)+f'(a)(x-a)}$$

$a=2$

$f(2)=8$

$f'(2)=12$

$L(x)=8+12(x-2)$ at $x=2.01$

Exp) Use linear approximation to estimate $\tan\left(\frac{\pi}{4}+0.01\right)$

$$L(x)=f(a)+f'(a)(x-a)$$

$f(x)=\tan x$

$a=\frac{\pi}{4}$ (known value)

$f(a)=f\left(\frac{\pi}{4}\right)=\tan\left(\frac{\pi}{4}\right)=1$

$f'(x)=\sec^2 x=(\sec x)^2$

$f'\left(\frac{\pi}{4}\right)=2=(\sqrt{2})^2$

$\sec x=\frac{1}{\cos x} \Rightarrow \sec\left(\frac{\pi}{4}\right)=\frac{1}{\cos\left(\frac{\pi}{4}\right)}=\frac{1}{\frac{1}{\sqrt{2}}}=\sqrt{2}$

$$L(x)=1+2\left(x-\frac{\pi}{4}\right)$$

$$\begin{aligned}\tan\left(\frac{\pi}{4}+0.01\right) &\approx L\left(\frac{\pi}{4}+0.01\right)\\ &\approx 1+2\left(\cancel{\frac{\pi}{4}}+0.01-\cancel{\frac{\pi}{4}}\right)\\ &\approx 1+2(0.01)\\ &\approx 1+0.02\\ &\approx \boxed{1.02}\end{aligned}$$

Exp) Use tangent line approximation to estimate $\sqrt{3.9}$

$f(x) = \sqrt{x}$

$L(x) = f(a) + f'(a)(x-a)$

$a = 4$ (known value, easy to compute)

$f(4) = \sqrt{4} = 2$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$

$a = 4$

$L(x) = f(4) + f'(4)(x-4)$

$L(x) = 2 + \dfrac{1}{4}(x-4)$

$\boxed{x_1 = 3.9}$

$L(3.9) = 2 + \dfrac{1}{4}(3.9-4)$

$\sqrt{3.9} \simeq 2 + \dfrac{1}{4}(-0.1)$

$\sqrt{3.9} \simeq \underset{(40)}{2} - \dfrac{1}{40}$

$\sqrt{3.9} \simeq \dfrac{79}{40} \simeq 1.975$

Differentials

$f'(x)$ Lagrange Notation

$\frac{df}{dx}$ Leibniz Notation

If $y=f(x)$, then the differential of y at $x=a$ is: $dy = f'(a)\cdot dx$

$dx \rightarrow$ differential of x (Δx)
$dy \rightarrow$ differential of y (Δy)

small change in x or y

Exp) Find differential of $f(x)=y=x^3$ at $x=2$.

$dy = f'(a)\cdot dx$

$a=2$

$f(x)=x^3 \Rightarrow f'(x)=3x^2$

$\Rightarrow f'(2)=3\cdot 2^2$

$=12$

$dy = f'(2)\cdot dx$

$\underset{\Delta y}{dy} = 12\cdot \underset{\Delta x}{dx}$

Exp) Find differential $d(\underbrace{x\cdot \cos x}_{y})$

$dy = ?$

$$dy = f'(x)\cdot dx$$

$f(x) = y = x\cdot \cos x$

$f'(x) = 1\cdot \cos x + x\cdot(-\sin x)$

$$dy = (\cos x - x\cdot \sin x)\cdot dx$$

Error Propagation

($dy \to \Delta y$, $dx \to \Delta x$)

Linear approximation can be used to study propagation of error, which describes error that accummulate from other errors in an approximation (imperfect measurements /lack of precision in the tools used may lead to errors in calculations)

Def: Let x_0 be the measured value of a quantity x

Δx be the error in measurement

$x_0 + \Delta x$ be the exact value of x

$$\Delta f = f(x_0 + \Delta x) - f(x_0)$$

$$\Delta x = (x_0 + \Delta x) - (x_0)$$

$$\Delta f \text{ (propagated error at } x_0) \approx \underbrace{|f'(x_0)|}_{|f'(x_0)| \approx \frac{\Delta f}{\Delta x}} \cdot \Delta x$$

% error is: $100\left(\frac{\Delta f}{f}\right)\%$

Exp) You measured the side of a cube as 10 cm. long. You concluded that the volume is $10^3 = 1000\ cm^3$.
If your original measurement of the side is accurate within 2%; approximately how accurate is your calculation of the volume?

$\Delta x \to$ error in the measurement of side

$x_0 \to$ measured length of side

$\Delta x = \mp 2\% \times 10 = \mp 0.2$ cm.

$\Delta V = ?$

$$V(x) = x^3$$
$$V'(x) = 3x^2$$
$$V'(10) = 3 \cdot 10^2 = 300$$

$$\Delta f \approx |f'(x_0)| \cdot \Delta x$$
$$\Delta V \approx |V'(10)| \cdot \Delta x$$
$$\Delta V \approx 300(\mp 0.2)$$
$$\Delta V \approx \mp 60\ cm^3$$

$V = 1000\ cm^3$, we can be off by $\mp 60\ cm^3$

% error is $100\left(\frac{\Delta V}{V}\right)\% = 100 \cdot \frac{60}{1000} = 6\%$

Marginal Analysis

Marginal analysis is concerned with estimating the effect on quantity (cost, revenue, profit) when the production is changed by a unit.

$C(x) \rightarrow$ cost of producing x units

$MC(x) \rightarrow$ marginal cost (additional cost of producing 1 more unit)

$MC(x) = C(x+1) - C(x)$ exact

$MC(x) \approx C'(x)$ approximation

$p(x) \rightarrow$ demand function (market price per unit)

$R(x) = p(x) \cdot x$ $\rightarrow$ revenue obtained from producing x units

$MR(x) = R(x+1) - R(x)$ exact

$MR(x) \approx R'(x)$ approximation

$MC(x), MR(x) \rightarrow$ additional cost, revenue from the $(x+1)^{th}$ item.

Exp) A manufacturer models the total cost (in $) of a particular item by:

$$C(x) = \frac{1}{8}x^2 + 3x + 98$$

and the price per item (in $) by:

$$p(x) = \frac{1}{3}(75 - x)$$

where x is the # of items produced $(0 \leq x \leq 50)$

a) Find the marginal cost & marginal revenue

b) Use marginal cost to estimate the cost of producing the 9th item. What's the actual cost.

c) Use marginal revenue to estimate the revenue of selling the 9th item. What's the actual revenue.

$$C(x) = \frac{1}{8}x^2 + 3x + 98$$

and the price per item (in \$) by:

$$p(x) = \frac{1}{3}(75 - x)$$

a) $MC(x) = C'(x) \cdot \Delta x \qquad \Delta x = 1$

$$= \frac{1}{8} \cdot 2x + 3 = \underbrace{\frac{x}{4} + 3}$$

$$MR(x) = [p(x) \cdot x]' = \left[\frac{1}{3}(75 - x) \cdot x\right]'$$

$$= \left[25x - \frac{1}{3}x^2\right]'$$

$$MR(x) = 25 - \frac{2}{3} \cdot x$$

b) $C'(8) = \frac{8}{4} + 3 = \$5$ (approx. Change in cost as x increases from 8 to 9)

$$\overbrace{\Delta C = C(9) - C(8)}$$

$$= 5.125 \approx \$5.13$$

exact

c) $R'(8) = 25 - \frac{2}{3} \cdot 8 \approx \$19.67$

$\Delta R = R(9) - R(8) = \$19.33$