4.1. Extreme Values of a Continuous Function

Def: Absolute Maximum.

Let f be a function defined on an interval I that contains the number c. The $f(c)$ is the absolute maximum of f on D if $f(c) \geq f(x)$ for all x in D.

Def: Absolute Minimum

$f(c)$ is the abs. min of f on D if $f(c) \leq f(x)$ for all x in D.

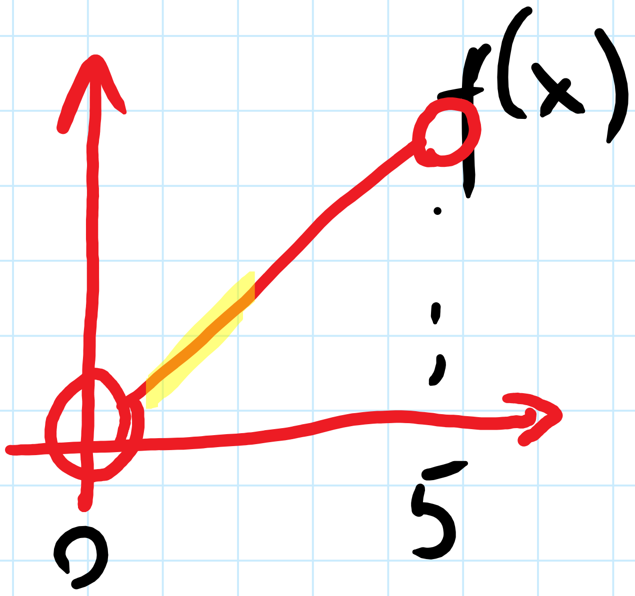
Absolute Extrema (Abs. Values) means Abs. min and Abs. max

Plural of Extremum is Extrema

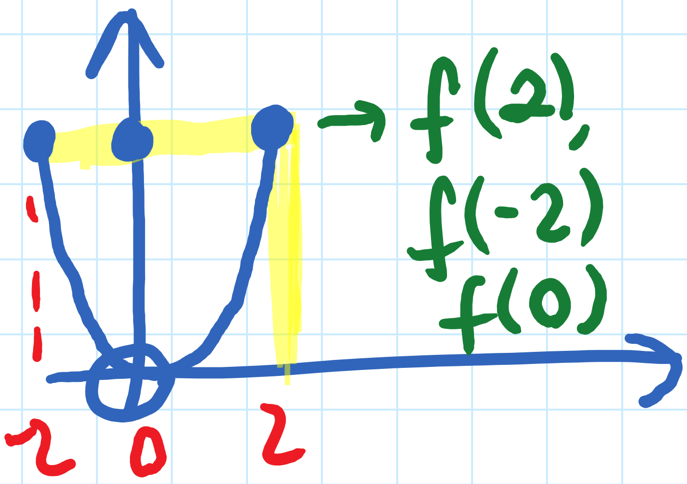
Extreme Value Theorem (EVT)

Existence Theorem

A function f has both an absolute minimum and an absolute maximum on any closed, bounded interval $[a,b]$ where it's continuous.



$(0,5)$ continuous function but w/ no abs. extrema

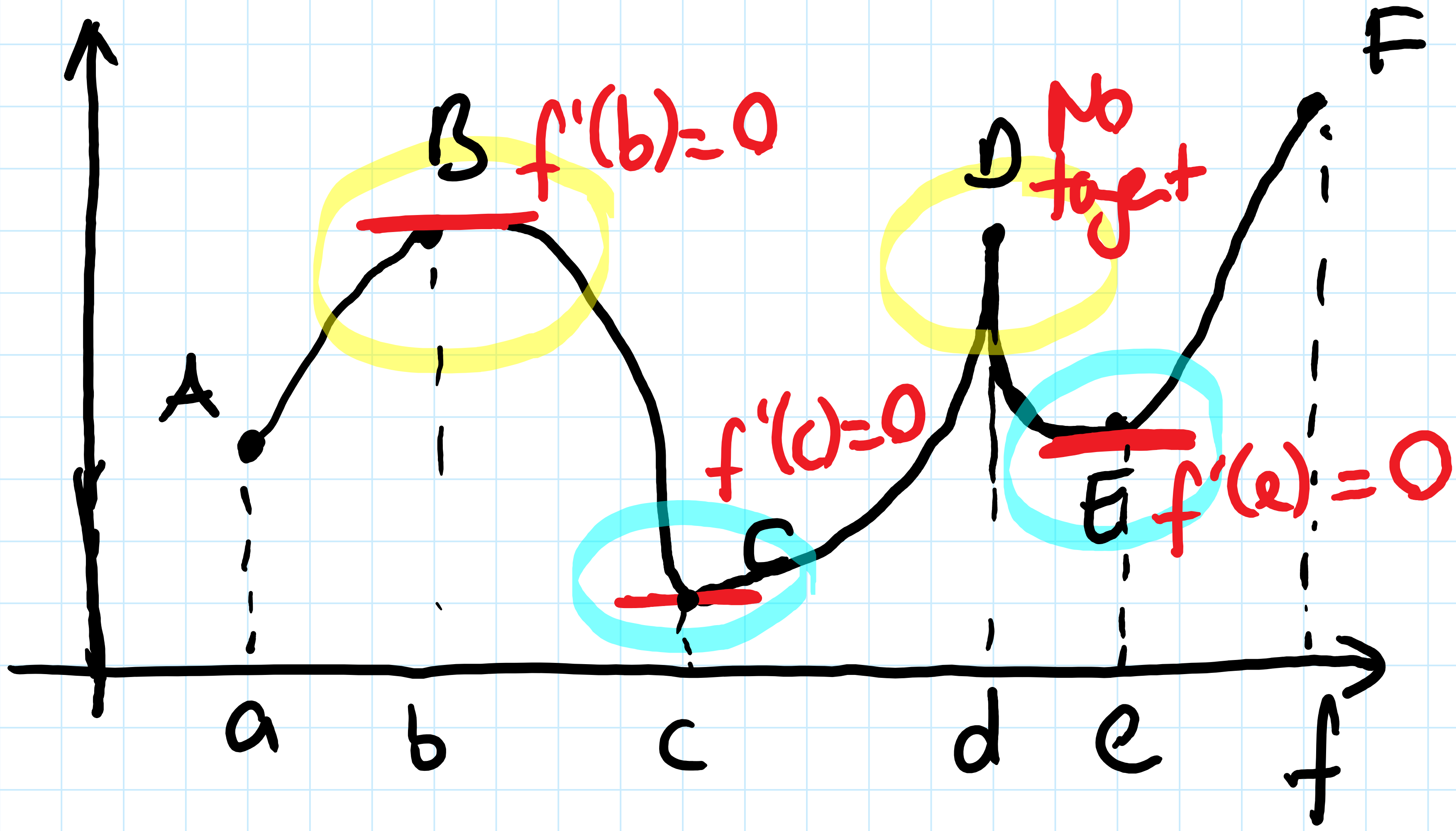


$[-2,2]$ point of discontinuity at $x=0$

$f(x)$ has an abs. max. at $x=0,-2,2$

however, the function doesn't have an abs. min.

Extreme values of a continuous function



$[a, f]$

peak points → B, D

valley points → C, E

(peak, valley points tells us the relative extrema)

The terms Relative and local are used interchangeably

The terms absolute and global are used interchangeably

Relative Maximum: A function f is said to have relative (local) max. at point c if $f(c) \geq f(x)$ for all x in an open interval containing c.

Relative Minimum: ___ ___ ___ ___ ___ ___ ___ if $f(c) \leq f(x)$ for all x in an open interval containing c.

Relative Extrema: Relative min and relative max are collectively called relative extrema.

Critical Number/Critical Point

f is $\boxed{\text{defined}}$ at c and either $f'(c) = 0$ or $f'(c)$ doesn't exist.

Then, the number c is called a critical number of f, and $P(c, f(c))$ is called the critical point on the graph of c.

Exp) Finding Critical Numbers $(x=c)$

a) $f(x)=4x^3-5x^2-8x+20$

C.N: $f'(x)=0$, ~~$f'(x)$ DNE~~

$f'(x)=12x^2-10x-8=0$

$=2(6x^2-5x-4)=0$

CMSA

$3x \quad -4$

$2x \quad 1$

$-8x+3x=-5x$

$=2\underbrace{(3x-4)}_{0}\underbrace{(2x+1)}_{0}=0$

Critical numbers are:

$x=\frac{4}{3}$, $x=-\frac{1}{2}$

Exp) Finding Critical Numbers $(x=c)$

b) $f(x)=\dfrac{e^x}{x-2}$

$$f'(x)=\frac{e^x(x-2)-e^x\cdot 1}{(x-2)^2}=\frac{e^x(x-2-1)}{(x-2)^2}=\frac{e^x(x-3)}{(x-2)^2}$$

$f'(x)=0$

$e^x(x-3)$

$\cancel{e^x=0}$ ~~never!~~, $x-3=0$

$\boxed{x=3}$

$f'(x)$ DNE

div. by 0

$(x-2)^2=0$

$\cancel{x=2}$

domain of $f(x)$: $(-\infty,2)\cup(2,\infty)$

Since $x=2$ is NOT in the domain of the original function, $x=2$ is NOT a critical #.

The only critical # for $f(x)$ is $x=3$.

Exp) Find critical #s

$x^{\frac{1}{2}} \cdot x^{1} = x^{3/2}$

$f(x) = 2\sqrt{x} \cdot (6-x) = 2 \cdot x^{\frac{1}{2}}(6-x) = 12x^{\frac{1}{2}} - 2x^{3/2}$

$f'(x) = 0$

$f'(x)$ DNE

$f'(x) = \left[12x^{\frac{1}{2}} - 2x^{\frac{3}{2}}\right]'$

$= 12 \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}} - 2 \cdot \frac{3}{2} \cdot x^{\frac{1}{2}}$

$= 6 \cdot x^{-\frac{1}{2}} - 3 \cdot x^{\frac{1}{2}} = \frac{6}{\sqrt{x}} - \frac{3 \cdot \sqrt{x}}{1} (\sqrt{x})$

$= 3x^{-\frac{1}{2}}\left[2 - x^{1}\right]$

$= \frac{3}{\sqrt{x}}(2-x) = \frac{3(2-x)}{\sqrt{x}}$

$= \frac{6 - 3 \cdot x}{\sqrt{x}}$

$= \frac{3(2-x)}{\sqrt{x}}$

$= 0$

$3(2-x) = 0$

$\boxed{x = 2}$ ✓

DNE

$\sqrt{x}$ in the denominator

$f'(0)$ DNE

$\boxed{x = 0}$ ✓

Domain of $f(x)$ is $[0, \infty)$ $(\sqrt{0} = 0)$

Exp) Critical Numbers and Critical Points

$$f(x) = (x-1)^2 \cdot (x+2)$$

$f'(x) = 0$ $\qquad$ $f'(x) \to DNE$

$$f'(x) = 2(x-1)^1(x+2) + (x-1)^2 \cdot 1$$
$$= (x-1)\left[2(x+2) + (x-1)\right]$$
$$= (x-1)(2x+4+x-1)$$
$$= (x-1)(3x+3) = 3(x-1)(x+1)$$

$$f'(x) = 0 = 3\underbrace{(x-1)}_{0}\underbrace{(x+1)}_{0} \qquad x = 1, \quad x = -1$$

~~$f'(x) \to DNE$~~ no x value

critical points are $(c, f(c))$

x	$f(x) = (x-1)^2(x+2)$
1	0
-1	$(-1-1)^2(-1+2)$ $(-2)^2(1) = 4$

Don't plug c in the $f'(x)$

critical points are $(1, 0)$ and $(-1, 4)$

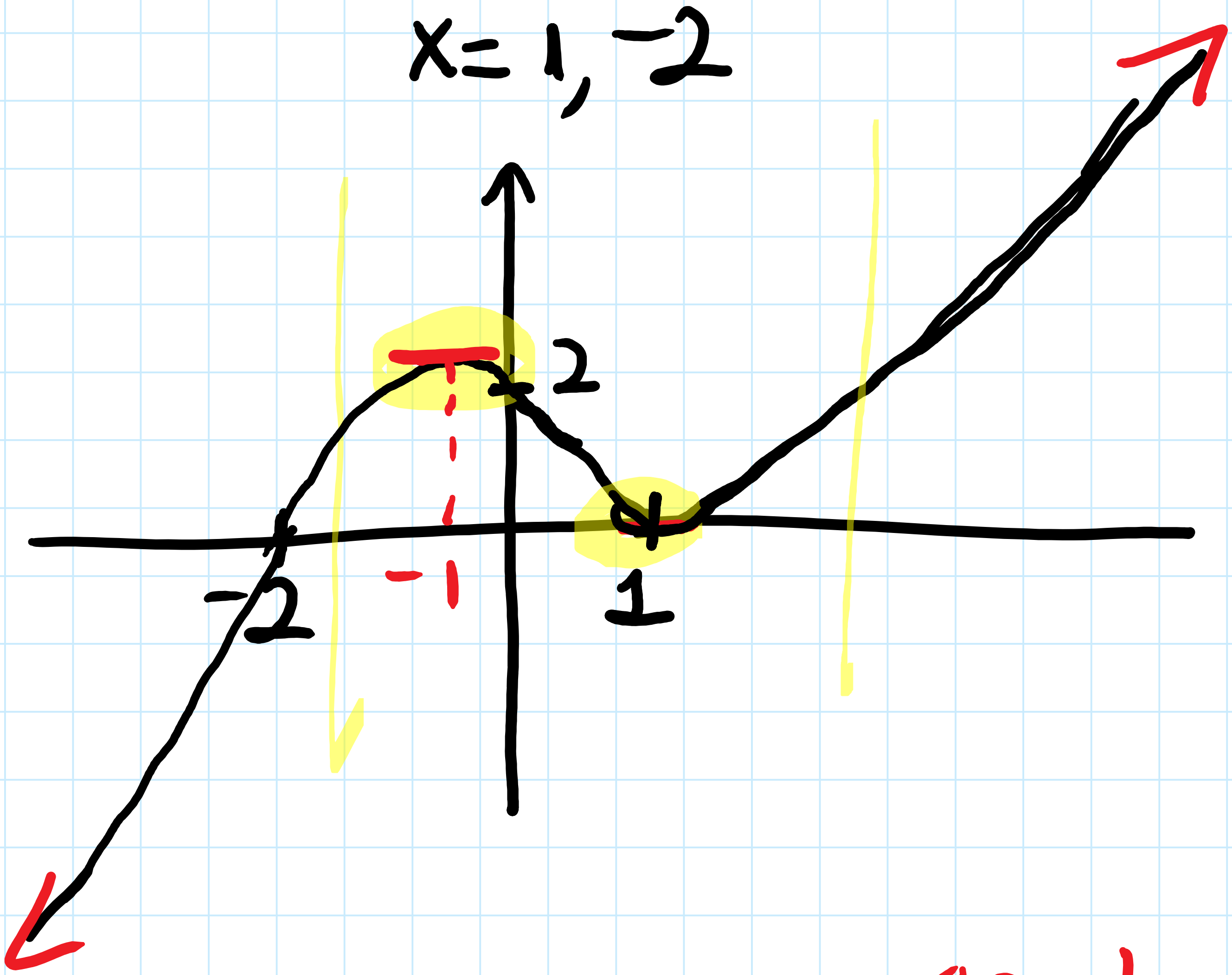
Precalc Observations

$f(x) = (x-1)^2 \cdot (x+2)$

L.T: $1x^3$

zeros are $f(x) = 0$

$x = 1, -2$



rise R

fall L

cross x-axis at $x = -2$

touches x-axis at $x = 1$

critical # $x = 1, -1$

Critical Number Theorem (CNT)

If a continuous function f has a relative extremum at c

Then c must be a critical number of f.

C.N.T. doesn't say that a relative extremum must occur at each critical #.

Procedure for Finding Absolute Extrema

To find the absolute extrema of a f on $[a,b]$

Step1) Compute $f'(x)$ and find all critical numbers of f on $[a,b]$

Step2) Evaluate f at the endpoints a and b AND at each critical number c

Step3) Compare the values ($f(x)$) in step 2. The max (largest) value of f is the Abs. Max. The min (smallest) value of f is the Abs. Min.

Exp) Find the absolute extrema of the function defined by the equation $f(x)=x^4-2x^2+3$ on the closed interval $[-1, 2]$

Step 1) $f'(x)=4x^3-4x$

$f'(x)=0$

~~$f'(x)$ DNE~~

no value of x that makes $f'(x)$ DNE

$f'(x)=4x(x^2-1)=0$

$4x=0$ $\quad x^2-1=0$

$x=0$ $\quad x=\mp 1$

Critical #s are $x=0, 1, -1$ (c)

$[-1,2]$

$[a, b]$

Step2)

x	$f(x)=x^4-2x^2+3$
0	$f(0)=3$
1	$f(1)=1-2+3=2$
-1	$f(-1)=(-1)^4-2(-1)^2+3=1-2+3=2$
2	$f(2)=2^4-2\cdot 2^2+3=16-8+3=11$

Step3) Compare ALL $f(c)$, $f(a)$, $f(b)$

Abs. maximum is 11 (at $x=2$)

Abs. min is 2 (at $x=1,-1$)