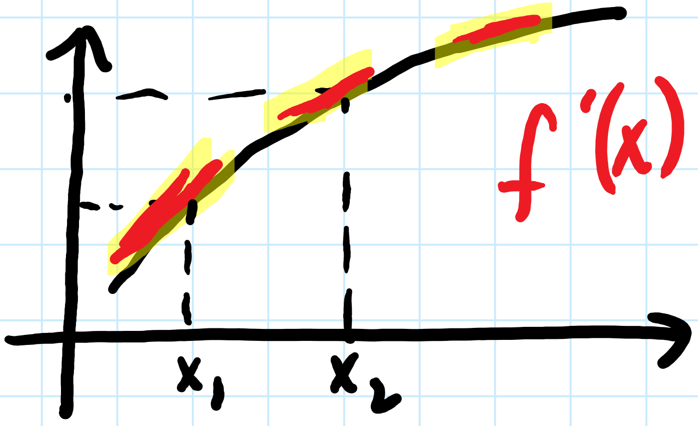
4.3. Using Derivatives to Sketch the Graph of a Function.

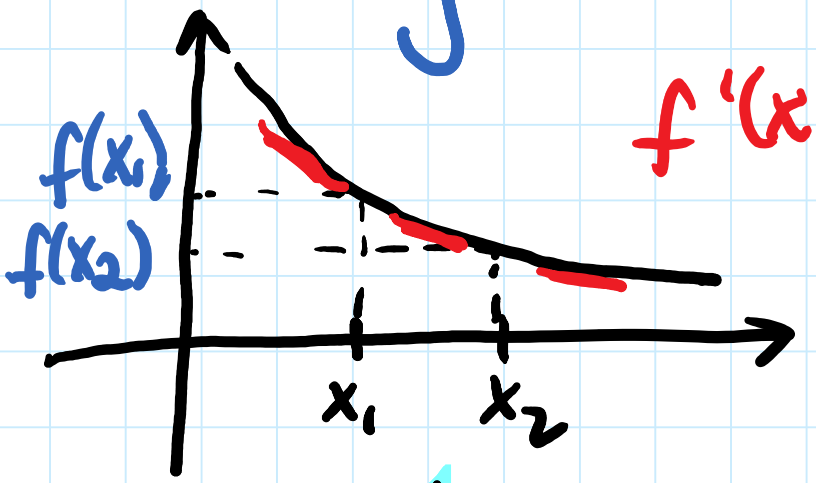
Goal: Use $f'(x)$ and $f''(x)$ to determine the graph of $f(x)$.

Terminology:

Increasing / Decreasing



$f'(x) > 0$



$f'(x) < 0$

$x_1 < x_2$

$f(x_1) < f(x_2)$

$f'(x) > 0$

$f(x)$ increasing

$x_1 < x_2$

$f(x_1) > f(x_2)$

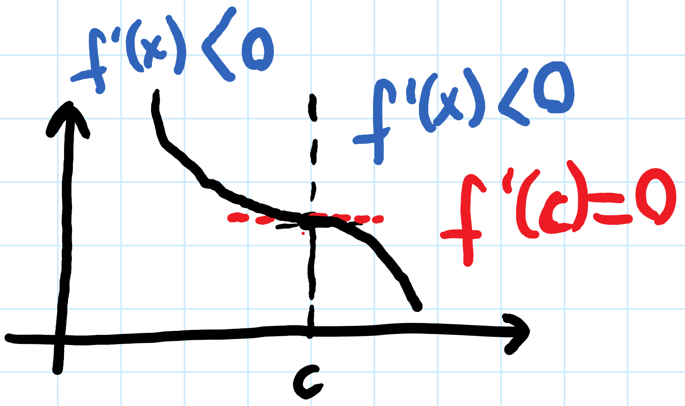
$f'(x) < 0$

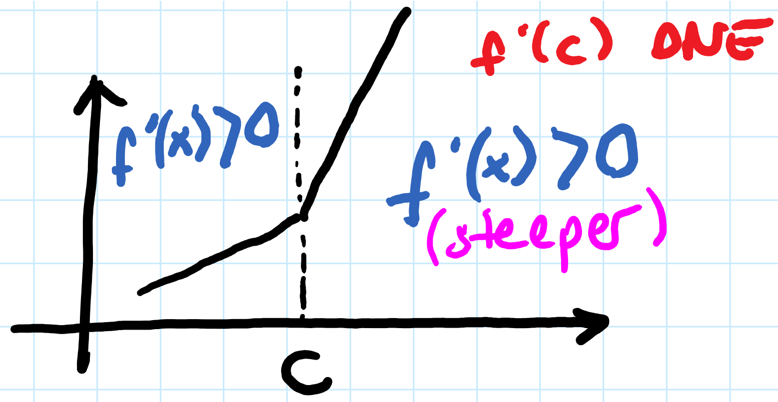
$f(x)$ decreasing

Relative / Local Extreme Values

Critical # (c.) can be only inside the f.

Local/relative extreme values can occur ONLY in the interior of the function's domain.





Although c is a critical # ($f'(c)=0$ or DNE); there's no sign change for $f'(x)$ at $x=c$; therefore; there's No local extrema.

First Derivative Test

Step 1: Find all critical #s of a continuous function f defined on (a, b).

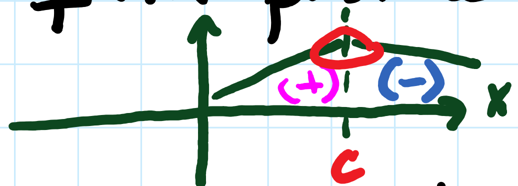
$f'(x)=0$ or $f'(x)$ DNE

critical #s $\rightarrow c$

Step 2: Classify each critical #(c) as follows:

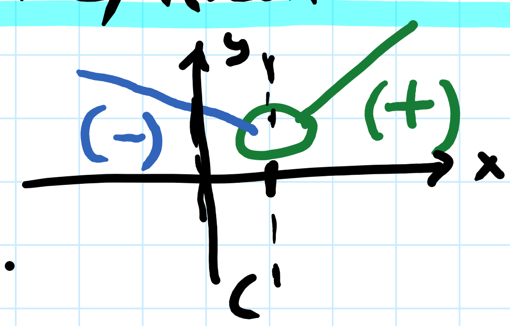
a) Point $(c, f(c))$ is a relative/local max:

If $f'(x)$ changes sign from positive to negative at $x=c$

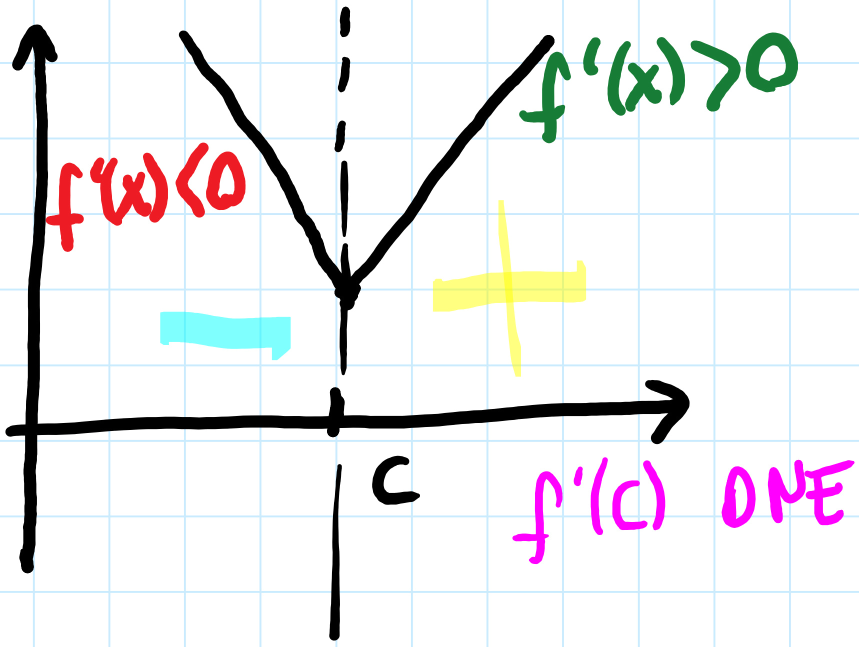


b) Point $(c, f(c))$ is a relative/local min:

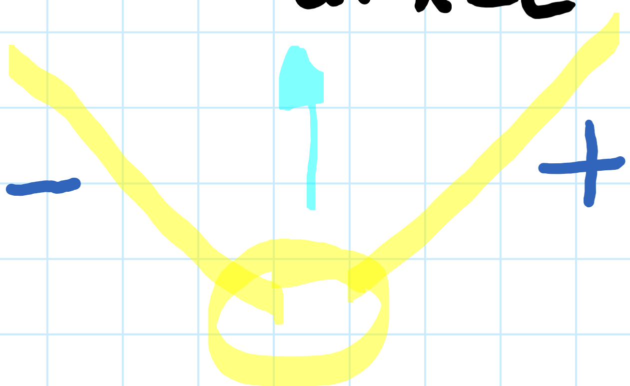
If $f'(x)$ changes sign from negative to positive at $x=c$.



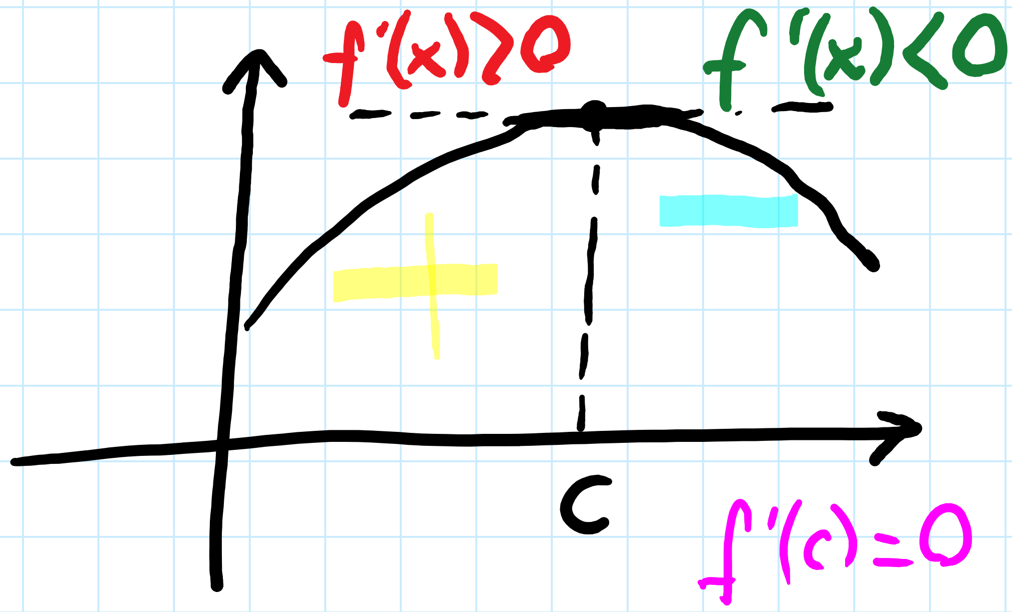
c) if $f'(x)$ doesn't change sign at $x=c$.



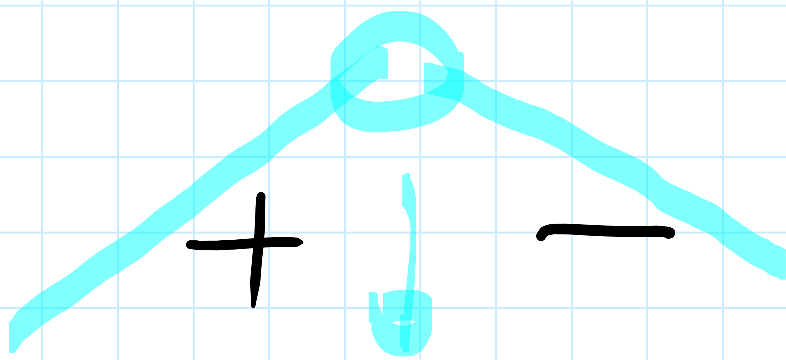
f'(x) changes sign at x=c



f(x) has a local min at x=c



f'(x) changes sign at x=c



f(x) has a local max. at x=c

Exp) Find local min/max of $f(x)=x^3-3x^2-9x+1$

Find where $f(x)$ is increasing/decreasing.

Find critical #s when $f'(x)=0$ or $f'(x)$ DNE

none

$f'(x)=3x^2-6x-9$

$=3(x^2-2x-3)$

$-3 \quad +1$

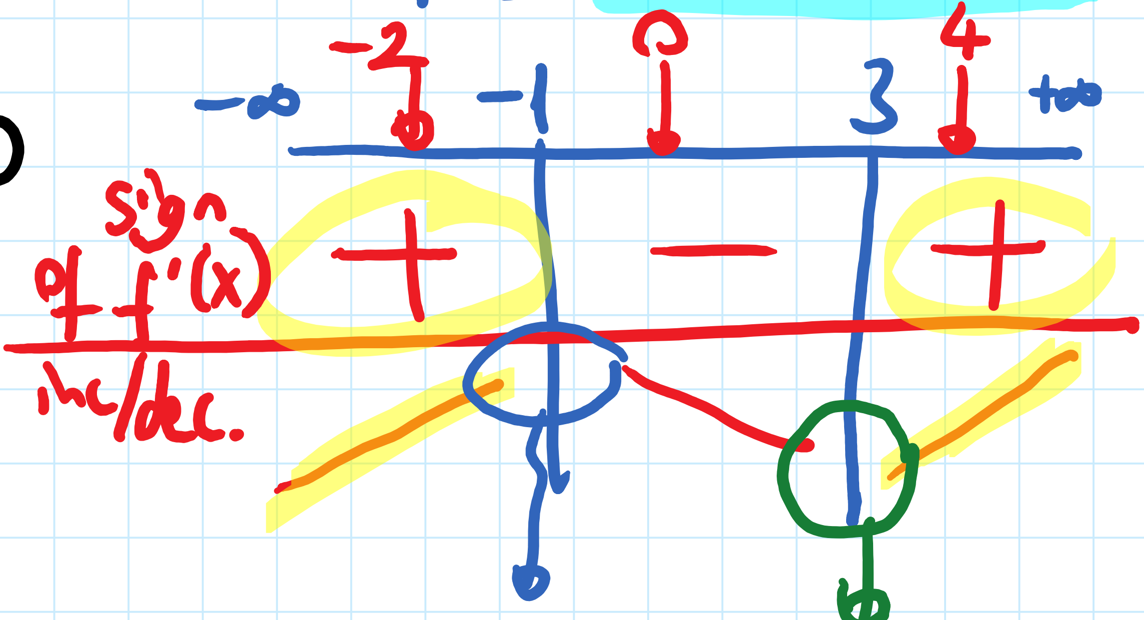
$= 3\underbrace{(x-3)}_{0}\underbrace{(x+1)}_{0} = 0$

critical #s: $x=3, x=-1$

cutpoints

Sign chart for $f'(x)$

$f'(x)=\boxed{3(x-3)(x+1)}$



test points: $x=-2, 0, 4$

$f'(-2)=3(-2-3)(-2+1)=3(-5)(-1)$ $+$

$f'(0)=3(0-3)(0+1)=3(-3)(1)$ $-$

$f'(4)=3(4-3)(4+1)$ $+$

$x=-1$ local max

$x=3$ local min

$f(x)$ has a local max at $x=-1$ point $(-1, f(-1))$

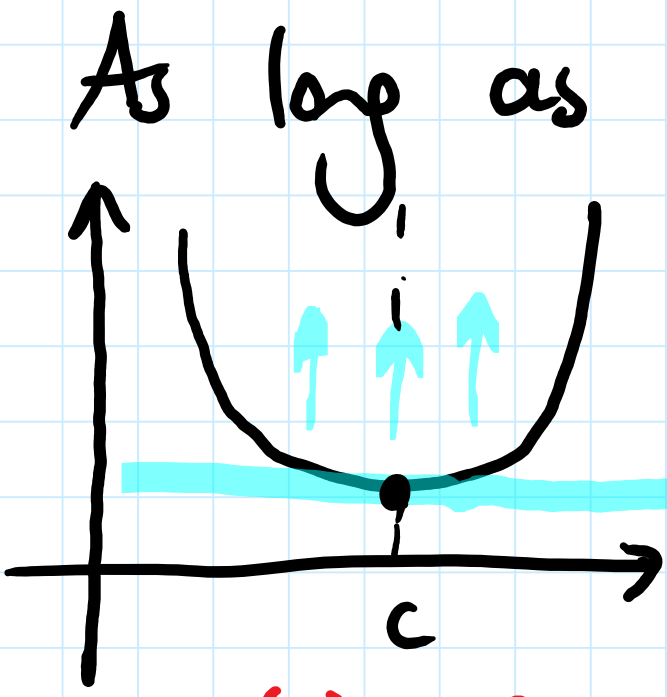
$f(x)$ has a local min at $x=3$ $(3, f(3))$

$f(x)$ is increasing on $(-\infty,-1), (3,\infty)$

$f(x)$ is decreasing on $(-1,3)$

Second Derivative Test (for local min/max)

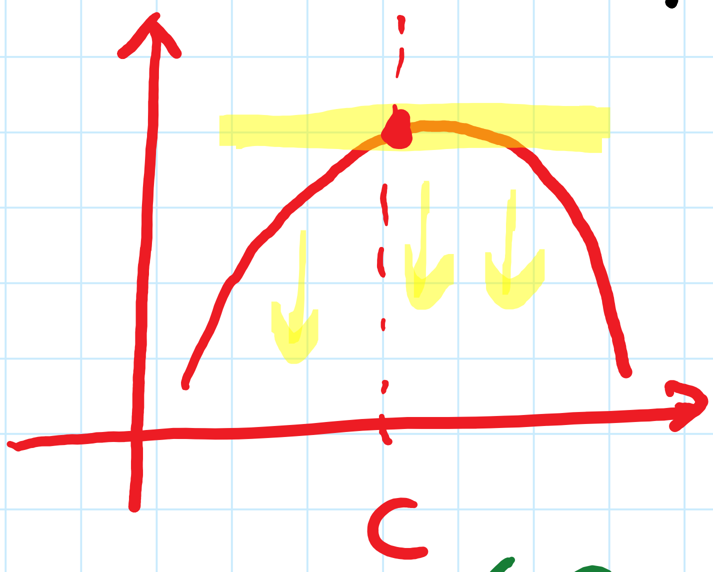
As long as $f''(x)$ is continuous and $f'(c)=0$



$f''(c)>0$

concave up

c is a local min



$f''(c)<0$

concave down

c is a local max.

Inflection Point

If the concavity of f changes at $x=c$, and f is continuous at $x=c$,

Then $(c, f(c))$ is an inflection point (P.I)

Inflection points can occur only where $f''(x)$ DNE or $f''(x)=0$.

Second-order critical #s: $f''(c)=0$ or DNE.

Second Derivative Test:

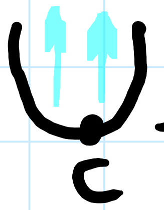
(To find local extreme values)

Step 1) Find all critical #s of continuous function f on (a,b) by:

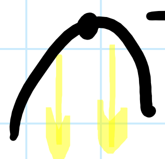
$f'(c)=0$ or $f'(c)$ DNE

($f''(x)$ is continuous at $x=c$)

Step 2) Classify the critical # as:

a) $f''(c) > 0$ 

$\rightarrow$ c is a local min

b) $f''(c) < 0$  $\rightarrow$ c is a local max

c) $f''(c) = 0$ the test fails / inconclusive

Procedure to graph $f(x)$

Step 1) $f(x)$ — points (x, y intercepts)
— V.A, H.A (Section 4.4)

Step 2) $f'(x)$ — first-order critical #s
— find where $f(x)$ is incr/decr.
— find if critical #s are actually local min/max

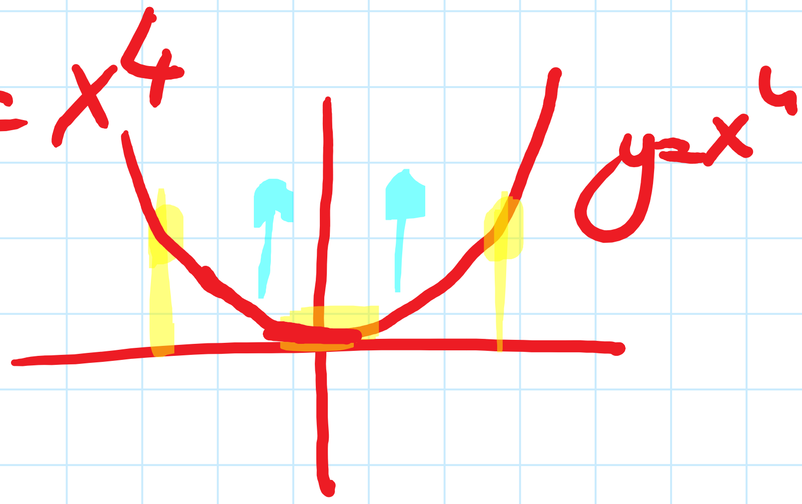
(construct a sign chart for $f'(x)$ by using critical #s and x-coord. of V.A as cutpoints)

Step 3) $f''(x)$ — second-order critical #s
— find where $f(x)$ concave up/down
— find if second-order critical #s are point of inflection (PoI) — (Is there a change in concavity at second-order critical #?)

(construct a sign chart for $f''(x)$ by using second-order critical #s and x-coord. of V.A as cutpoints)

Step 4) Sketch $f(x)$ by including important points

Exp) Find PoI for $f(x)=x^4$



$f'(x)=4x^3$ $\quad f'(x)=0$, $x=0$

$f''(x)=12x^2$

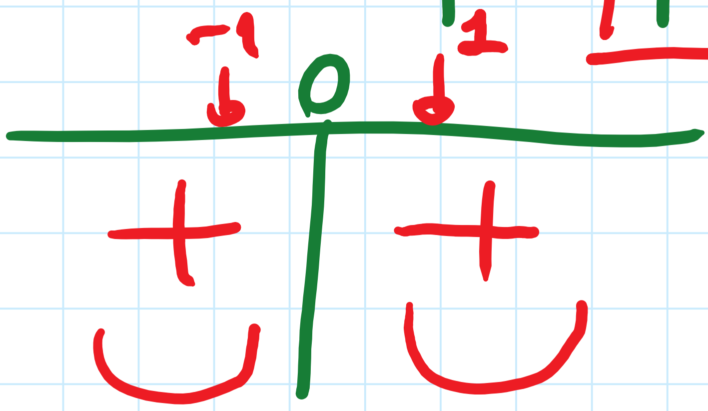
$f''(x)=0$ (second-order critical #s)

$12x^2=0 \Rightarrow \boxed{x=0}$

Is there a sign chart at $x=0$ for $f''(x)$

Let's see sign chart for $\boxed{f''(x)}$

$f''(x)=12x^2$



test points:

$x=-1, 1$

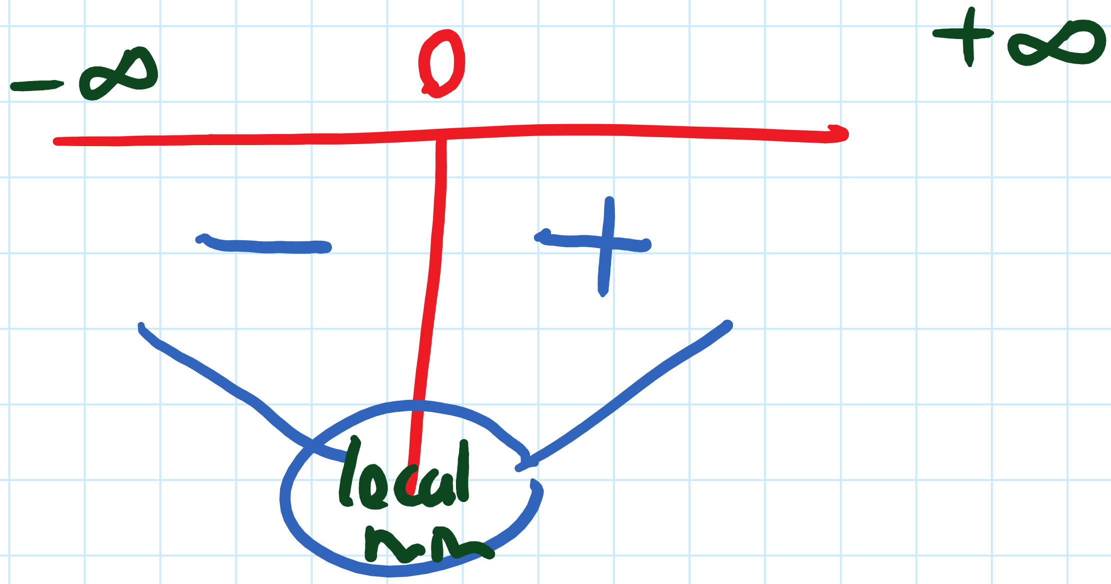
Since, concavity doesn't change (sign of $f''(x)$)

there's $\boxed{\text{NO PoI.}}$

Sign chart for $f'(x)=4x^3$

local min at $x=0$

local max → NONE



Exp: Graph $f(x) = x^4 - 4x^3 + 10$

Step 1) $f(x)$ $\quad x = 0 \rightarrow$ y-int: $f(0) = 10$ $\quad (0, 10)$

Step 2) $f'(x)$ $\quad f'(x) = 4x^3 - 12x^2$

$$= 4x^2 (x-3)$$

critical # are: when $f'(x) = 0$ or DNE

none

$$\boxed{f'(x) = 4\underbrace{x^2}_{0}\underbrace{(x-3)}_{0}}$$

$x = 0, 3$ are critical #s

construct a sign chart for $f'(x)$

cutpoints: $x = 0, 3$

testpoints: $x = -1, 1, 4$

$f'(-1) = 4(-1)^2(-1-3)$ $-$

$f'(1) = 4(1)^2(1-3)$ $-$

$f'(4) = 4(4)^2(4-3)$ $+$

	-1	0	1	3	4
sign of $f'(x)$	$-$		$-$		$+$
increase / decr.					

valley

local min at $x = 3$

$f(x)$ has a local min at $x = 3$

$f(x)$ is decreasing on $(-\infty, 3)$

$f(x)$ is increasing on $(3, \infty)$

3) $f''(x)$

$$f'(x) = 4x^3 - 12x^2$$

$$f''(x) = 12x^2 - 24x$$

$$= 12x(x-2)$$

Second-order critical #s are: $f''(x) = 0$ or DNE

none

$$f''(x) = 12x(x-2) = 0$$

$$\boxed{x = 0, 2}$$

Now, construct a sign-chart for $f''(x)$

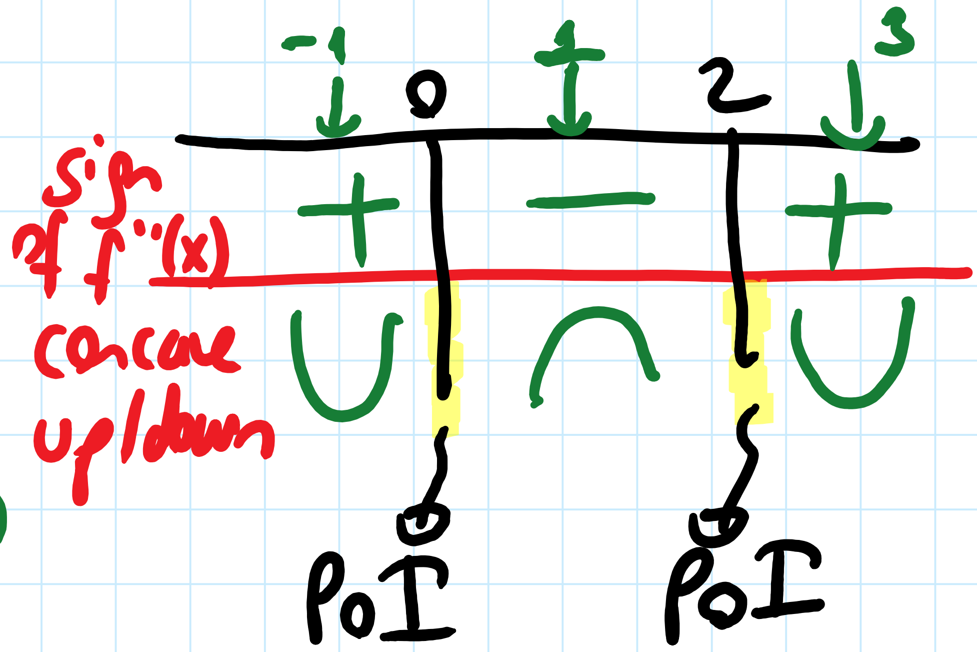
cutpoints: $x = 0, 2$

testpoints: $x = -1, 1, 3$

$f''(-1) = 12(-1)(-1-2) \Rightarrow (+)$

$f''(1) = 12(1)(1-2) \Rightarrow (-)$

$f''(3) = 12(3)(3-2) \Rightarrow (+)$

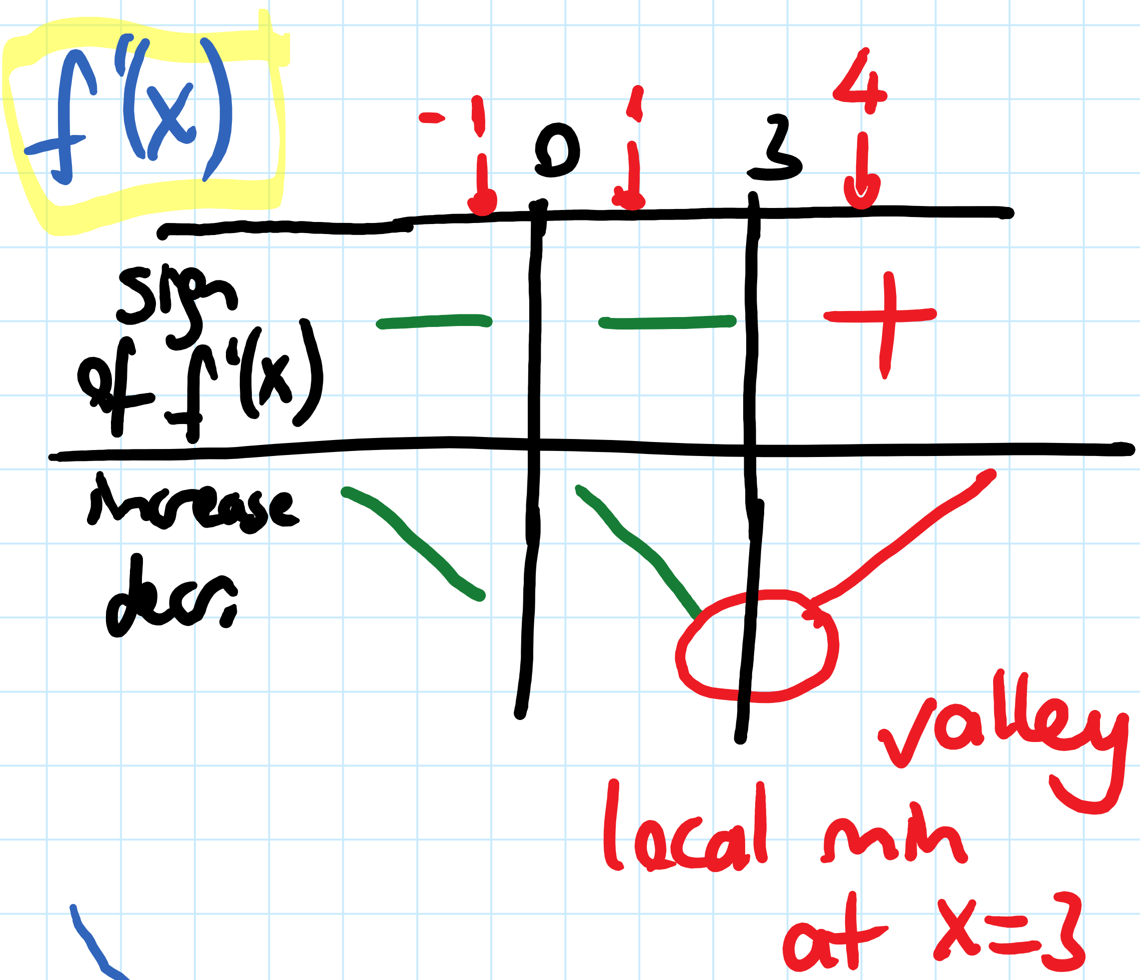


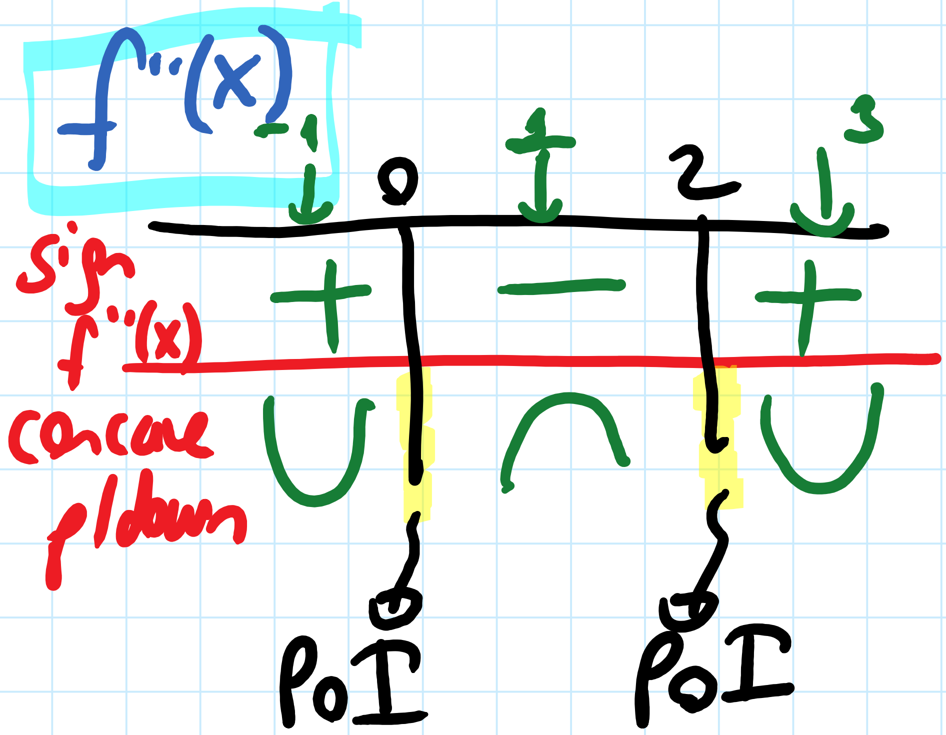
Since concavity changes at $x = 0, 2$

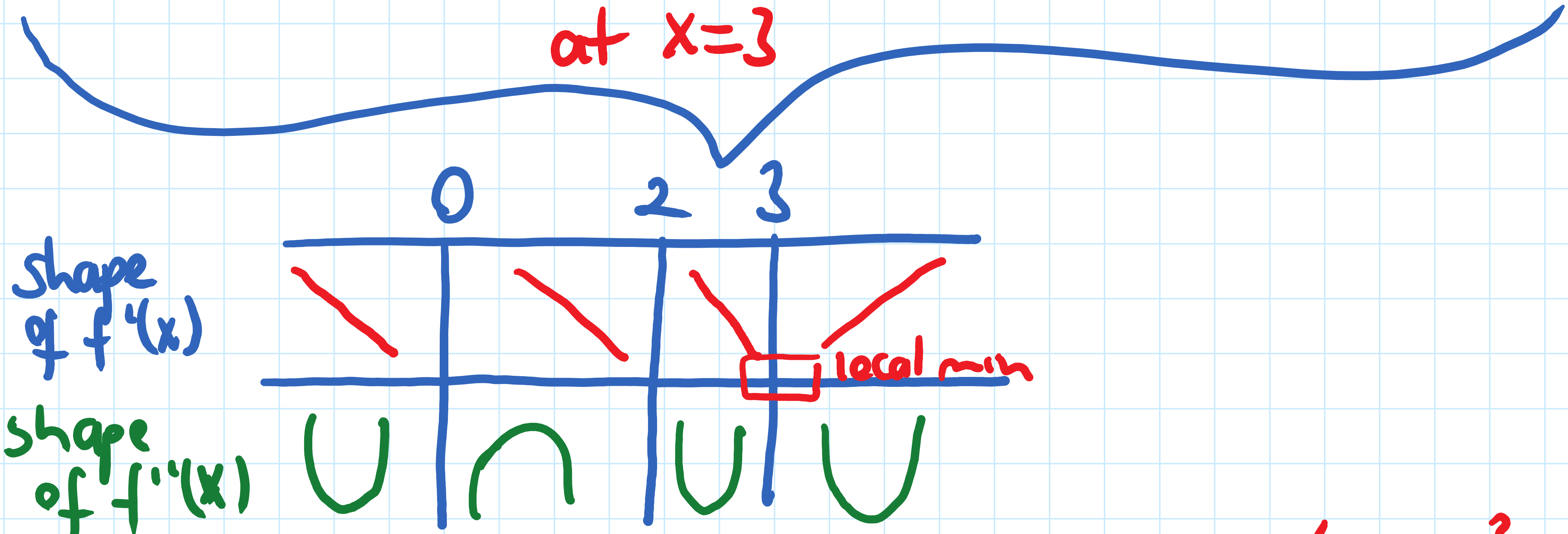
$x = 0$ and $x = 2$ are the x-coordinates of PoI

$(0, f(0))$, $(2, f(2))$

PoI







4) Important Points:

$f(x) = x^4 - 4x^3 + 10$
$= x^3(x-4) + 10$

y-int: $(0, 10)$

1st order critical points: $f'(x) = 0 \Rightarrow x = 0$

$f(3) = 3^3(3-4) + 10$
$f(3) = 27(-1) + 10 = -17$

$\boxed{(0,10);\ (3,-17)}$

2nd order critical points: $x=0, x=2$

Since the concavity changes at $x=0,2$

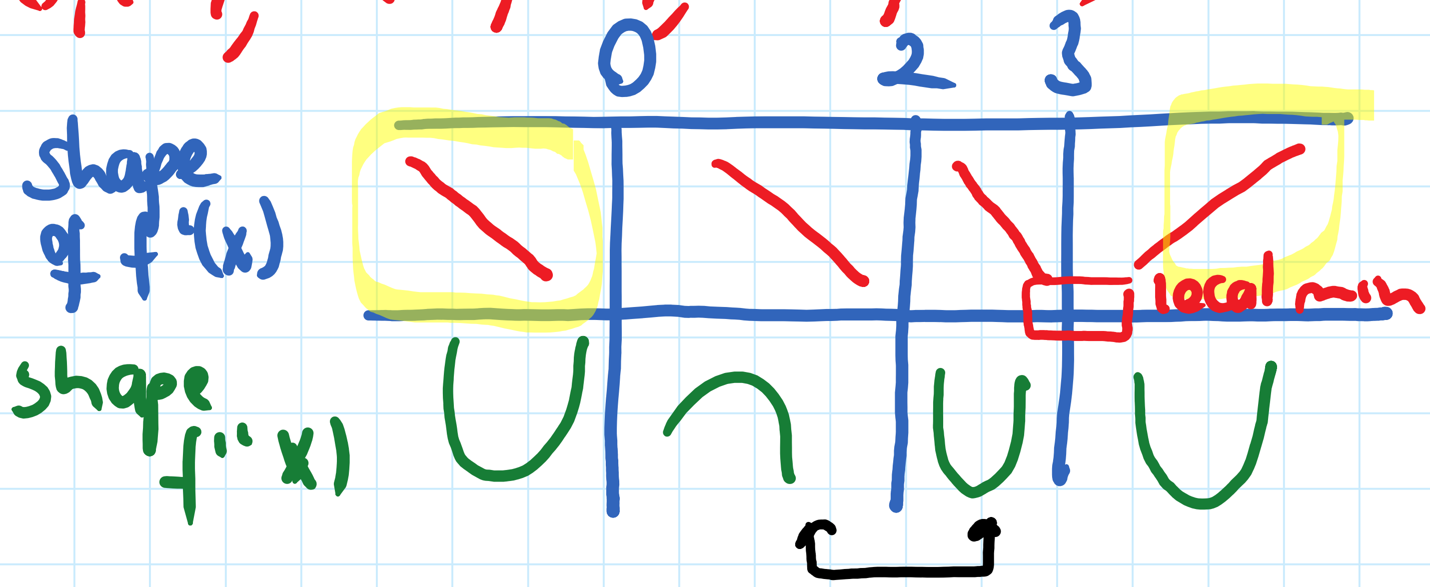
These points are also PoI at: $(0, f(0)), (2, f(2))$

$f(2) = 2^3(2-4) + 10 = 8(-2) + 10$
$f(2) = -6$

$\boxed{(0,10), (2,-6)}$

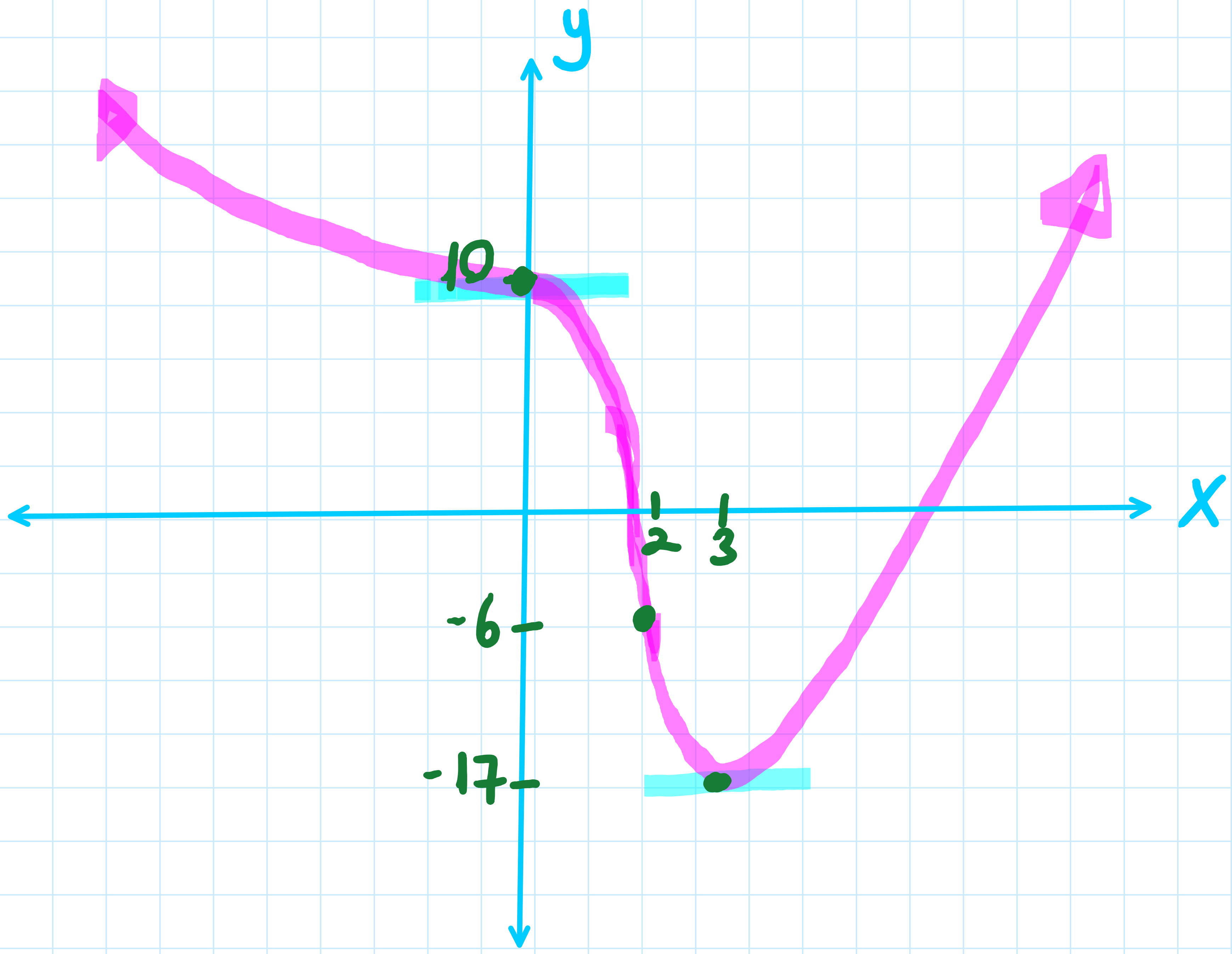
Put everything together to graph $f(x)$.

$(0,10)$, $(2,-6)$, $(3,-17)$



$x=3$, $x=0$ Horizontal Tangent

$x=3$ local min



Graph of $f(x)=x^4-4x^3+10$
(Not drawn to scale)