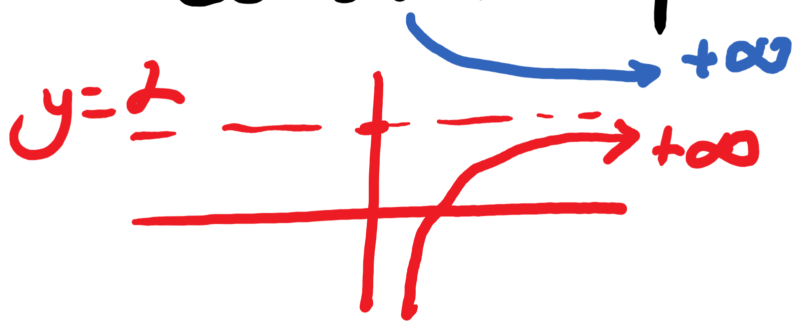
4.4. Curve Sketching with Asymptotes

Limits at Infinity

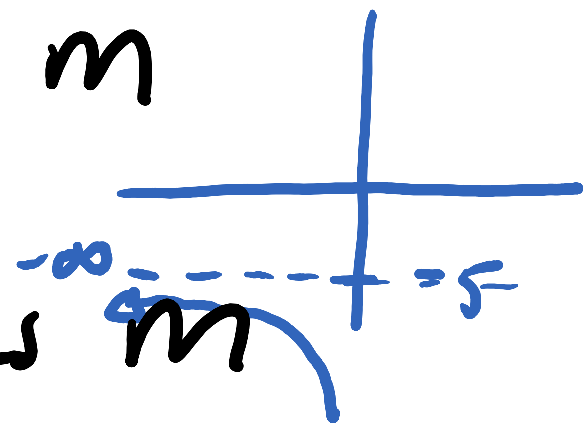
To indicate the "long run" behavior of a function, we write



$$\lim_{x\to\infty} f(x) = L$$

to show that $f(x)$ approaches L as x approaches $+\infty$ (increases w/out bounds).

We also write $\lim_{x\to -\infty} f(x) = m$



to show that $f(x)$ approaches m as x decreases without bounds.

Theorem (Special Limits at Infinity)

If A is any real #, r is a positive rational #, then $\lim_{x\to\infty} \frac{A}{x^r} = 0$

$$\lim_{x\to\infty} \frac{5}{x^{10}} = 0$$

If r is such that x^r is defined for $x<0$, then $\lim_{x\to -\infty} \frac{A}{x^r} = 0$

When evaluating a limit of the form:

$\lim_{x \to \mp\infty} \frac{p(x)}{q(x)}$ where p, q are polynomials,

it's often useful to divide both p, q by the highest power of x that occurs in the denominator. Apply Th. on Special Limits at Infinity.

Exp) Evaluate $\lim_{x \to \infty} \frac{3x^3 - 5x + 9}{5x^3 + 2x^2 - 7}$ → highest power of x is x^3.

recall: $\boxed{\lim_{x \to \infty} \frac{A}{x^r} = 0}$

$$\lim_{x \to \infty} \left(\frac{\frac{3x^3}{x^3} - \frac{5x}{x^3} + \frac{9}{x^3}}{\frac{5x^3}{x^3} + \frac{2x^2}{x^3} - \frac{7}{x^3}} \right)$$

$$\frac{\lim_{x \to \infty} \left(3 - \frac{5}{x^2} + \frac{9}{x^3} \right)}{\lim_{x \to \infty} \left(5 + \frac{2}{x} - \frac{7}{x^3} \right)} = \frac{\lim_{x \to \infty} (3) - \lim_{x \to \infty} \left(\frac{5}{x^2} \right) + \lim_{x \to \infty} \frac{9}{x^3}}{\lim_{x \to \infty} (5) + \lim_{x \to \infty} \frac{2}{x} - \lim_{x \to \infty} \frac{7}{x^3}}$$

$$= \frac{\lim_{x \to \infty} 3}{\lim_{x \to \infty} 5} = \frac{3}{5}$$

Exp) Evaluate $\lim_{x\to\infty} \sqrt{\dfrac{3x-5}{x-2}}$

$$\lim_{x\to\infty} \left(\frac{3x-5}{x-2}\right)^{1/2} = \left[\lim_{x\to\infty} \left(\frac{3x-5}{x-2}\right)\right]^{1/2}$$

highest power x is x in the denominator

$$= \left[\lim_{x\to\infty} \left(\frac{\frac{3x}{x} - \frac{5}{x}}{\frac{x}{x} - \frac{2}{x}}\right)\right]^{1/2}$$

$$= \left[\lim_{x\to\infty} \left(\frac{3 - \frac{5}{x}}{1 - \frac{2}{x}}\right)\right]^{1/2} = \left(\frac{\lim_{x\to\infty}(3) - \lim_{x\to\infty}\frac{5}{x}}{\lim_{x\to\infty}(1) - \lim_{x\to\infty}\frac{2}{x}}\right)^{1/2}$$

0 0

$$= \left[\frac{\lim_{x\to\infty}(3)}{\lim_{x\to\infty}(1)}\right]^{1/2} = \left[\frac{3}{1}\right]^{1/2} = \boxed{\sqrt{3}}$$

Exp) Evaluate $\lim\limits_{x\to-\infty} \dfrac{5x^3-2x}{x^2+1}$ → highest power of x is x^2

$$\lim_{x\to-\infty}\left(\frac{\frac{5x^3}{x^2}-\frac{2x}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}}\right)=\lim_{x\to-\infty}\left(\frac{5x-\frac{2}{x}}{1+\frac{1}{x^2}}\right)=\frac{-\infty}{1}=-\infty$$

Infinite Limits

∞ is NOT a number, so an infinite limit doesn't exist in the sense that limits were defined in Ch. 2. However, to distinguish "$\pm\infty$" limit from other ways for a limit not to exist (such as $\lim\limits_{x\to\infty}\cos x$ fails to exist by oscillation)

we write: $\lim\limits_{x\to c} f(x) = \mp\infty$

The limit statement: $\lim\limits_{x\to c} f(x) = +\infty$

means that f increases w/out bound as $x\to c$ from either side.

$\lim\limits_{x\to c} f(x) = -\infty$ means that f decreases w/out bound as $x\to c$ from either side.

Exp) Find $\lim_{x\to 2^-}\left(\frac{3x-5}{x-2}\right)$, $\lim_{x\to 2^+}\left(\frac{3x-5}{x-2}\right)$

$$\lim_{x\to 2^-}\left(\frac{3x-5}{x-2}\right) = \frac{\lim_{x\to 2^-}(3x-5)}{\lim_{x\to 2^-}(x-2)} = \frac{1}{\underbrace{\lim_{x\to 2^-}(x-2)}}$$

$$= \frac{1}{\text{"very tiny neg. \#"}} = \boxed{-\infty}$$

$$\lim_{x\to 2^+}\left(\frac{3x-5}{x-2}\right) = \frac{1}{\lim_{x\to 2^+}(x-2)} = \frac{1}{\text{"very tiny pos. \#"}} = \boxed{+\infty}$$

Graphs w/ Asymptotes

Vertical Asymptote: The line $x=c$ is a vertical asymptote (V.A) of the graph of f if either of the one-sided limits

$$\lim_{x\to c^-} f(x) \quad \text{or} \quad \lim_{x\to c^+} f(x) \quad \text{is infinite}$$

Horizontal Asymptote: The line $y=L$ is a horizontal asymptote (H.A.) of the graph of f is:

$$\lim_{x\to\infty} f(x)=L \quad \text{or} \quad \lim_{x\to-\infty} f(x)=M.$$

Simplify rational function first, then look for values that cause the denominator to be 0 (and the numerator not to be zero).

Exp) Find the V.A of $f(x)=\dfrac{x^2-4}{x-2}$

$$f(x)=\frac{(\cancel{x-2})(x+2)}{\cancel{x-2}} = x+2 \;;\; \boxed{x\neq 2}$$

$f(x)$ doesn't have a V.A.

$g(x)=x+2$

Exp) Graph a rational function $f(x)=\frac{3x-5}{x-2}$

Step1) $f(x)$

x-int: $y=0=f(x)=\frac{3x-5}{\cancel{x-2}}=0 \Rightarrow 3x-5=0$

$x=5/3$ $\boxed{\left(\frac{5}{3},0\right)}$

y-int: $x=0 \Rightarrow f(0)=\frac{3\cdot 0-5}{0-2}=\frac{-5}{-2}=\frac{5}{2}$ $\boxed{\left(0,\frac{5}{2}\right)}$

$\boxed{\text{V.A:}\quad x-2=0,\ x=2}$

($f(x)$ is NOT defined at $x=2$)

($f(x)$ can NOT cross V.A.)

H.A:

$$\lim_{x\to\infty}\left(\frac{3x-5}{x-2}\right)=\lim_{x\to\infty}\left(\frac{\frac{3x}{x}-\frac{5}{x}}{\frac{x}{x}-\frac{2}{x}}\right)=\lim_{x\to\infty}\left(\frac{3-\frac{5}{x}}{1-\frac{2}{x}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{3}{1}\right)=3$$

$$\lim_{x\to-\infty}\left(\frac{3-\frac{5}{x}}{1-\frac{2}{x}}\right)=3$$

H.A.

$$\boxed{\lim_{x\to\mp\infty} f(x)=3}$$

Step2) $f'(x) = \dfrac{-1}{(x-2)^2}$ ($f(x)$ is NOT defined at $x=2$)

Find first-order critical #s:

$\underbrace{f'(x)=0}_{\text{none}}$ or $\underbrace{\text{DNE}}_{x=2}$

V.A

$$f'(x) = \frac{-1}{(x-2)^2}$$

When constructing the sign chart for $f'(x)$ we must include x-coord. of V.A ($x=2$)

sign of $f'(x)$

$$\frac{-}{+} = -$$

	$-\infty$	2	$+\infty$
sign of $f'(x)$	$-$		$-$
incr/decr.	↘		↘

$f(x)$ is decreasing on $(-\infty, 2)$, $(2, \infty)$

∪

AND

$f(x)$ has no local min/max (NONE)

Step 3) $f''(x) = \dfrac{2}{(x-2)^3}$

$\boxed{f(x) \text{ is not defined at } x=2}$

$f''(x) = 0$ or DNE — none

testpoints: $x = 1, 3$

$f''(1) = \dfrac{2}{(1-2)^3} = \dfrac{+}{-} \; (-)$

$f''(3) = \dfrac{2}{(3-2)^3} = \dfrac{+}{+} \; (+)$

$f(x)$ is concave down on $(-\infty, 2)$

$f(x)$ is concave up on $(2, \infty)$

there's no point of inflection (PoI).

Sign chart for $f''(x)$

	1	2	3
sign of $f''(x)$	$-$		$+$
concave up/down	$\cap$ down		$\cup$ up

Is there a PoI at $x=2$?

NO!

sign chart for $f'(x)$

	$-\infty$	2	$+\infty$
sign of $f'(x)$	$-$		$-$
incr/decr.	↘		↘

sign chart for $f''(x)$

	1	2	3
sign of $f''(x)$	$-$		$+$
concave up	$\cap$		$\cup$

$$f(x) = \frac{3x-5}{x-2}$$

Step(4) important points:

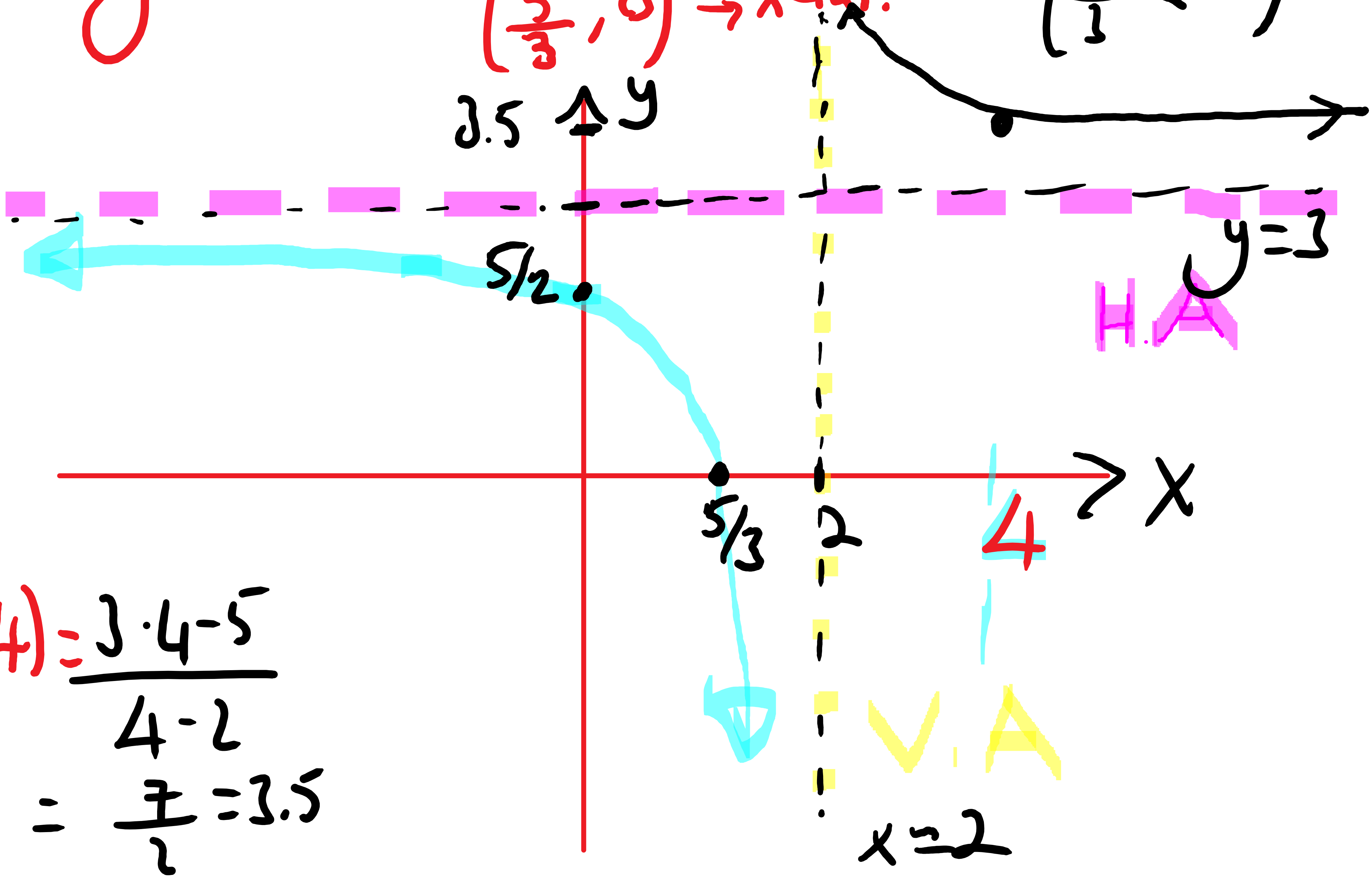
V.A: $x=2$

H.A: $y=3$

$f(0) = \frac{5}{2} \rightarrow \left(0, \frac{5}{2}\right) \rightarrow$ y-int.

$\left(\frac{5}{3}, 0\right) \rightarrow$ x-int.

$\left(\frac{5}{3} < 2\right)$



$$f(4) = \frac{3 \cdot 4 - 5}{4-2}$$

$$= \frac{7}{2} = 3.5$$