4.5. L'Hôpital's Rule

Goal: To evaluate indeterminate forms (limits that their value can't be determined without further analysis)

Theorem: Let f, g be differentiable functions w/ $g'(x) \neq 0$ on an open interval containing c. Suppose $\lim_{x \to c} \frac{f(x)}{g(x)}$ produces an indeterminate form $\boxed{\left(\frac{0}{0}, \frac{\infty}{\infty}\right)}$ and that $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$ where L is either a $\boxed{\text{finite \#, } +\infty \text{ or } -\infty.}$

Then $\lim_{x \to c} \frac{f(x)}{g(x)} = L$.

This applies to one-sided limits and to limits at ∞ $(x \to \pm\infty)$.

Do NOT use quotient rule for f/g

Justify the use of L.R. $\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$ and ... $\left(\frac{-\infty}{-\infty}, \frac{-\infty}{\infty}, \frac{\infty}{-\infty}\right)$

Exp1) Evaluate $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1.$

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{\text{"DSP"}}{=} \frac{\sin 0}{0} = \text{"}\frac{0}{0}\text{"}$$

$$\lim_{x\to 0} \frac{\cos x}{1} \overset{\text{"DSP"}}{=} \cos 0 = 1$$

Exp2) Evaluate $\lim\limits_{x \to 0} \frac{1-\cos x}{\sec x}$

Recall: $\sec x = \frac{1}{\cos x}$

$$\lim_{x\to 0} \frac{1-\cos x}{\sec x} \overset{\text{"DSP"}}{=} \frac{1-\cos 0}{\sec 0} = \frac{1-1}{\frac{1}{\cos 0}} = \frac{0}{\frac{1}{1}} = \frac{0}{1} = 0$$

What if we use L.R. w/out justifying?

$$\lim_{x\to 0} \frac{0+\sin x}{\sec x \cdot \tan x} \overset{\text{"DSP"}}{=} \frac{\sin 0}{\sec 0 \cdot \tan 0} = \frac{\cancel{\sin 0}}{\frac{1}{\cos 0} \cdot \frac{\cancel{\sin 0}}{\cos 0}} = \frac{1}{\frac{1}{\cos^2 0}} = 1$$

Wrong!

Exp3) Evaluate $\lim\limits_{x \to 1} \frac{e^{2x}-e^2}{\ln(x)}$

Recall: $\left(e^{g(x)}\right)' = e^{g(x)} \cdot g'(x)$

$$\lim_{x\to 1} \frac{e^{2x}-e^2}{\ln(x)} \overset{\text{"DSP"}}{=} \frac{e^2-e^2}{\ln 1} = \text{"}\frac{0}{0}\text{"}$$

$$\lim_{x\to 1} \frac{e^{2x}\cdot 2}{\frac{1}{x}} = \lim_{x\to 1} 2\cdot x\cdot e^{2x} \overset{\text{"DSP"}}{=} 2\cdot 1\cdot e^2 = \boxed{2e^2}$$

Exp4) Evaluate $\lim_{x\to\infty} \frac{2x^2+3x+1}{3x^2+5x-2}$ "$\frac{\infty}{\infty}$" → find the highest exponent of x, div. ALL

limits at ∞ (H.A.)

ln 4.4. $\lim_{x\to\infty} \dfrac{\frac{2x^2}{x^2}+\cancel{\frac{3x}{x^2}}^{\,0}+\cancel{\frac{1}{x^2}}^{\,0}}{\frac{3x^2}{x^2}+\cancel{\frac{5x}{x^2}}^{\,0}-\cancel{\frac{2}{x^2}}^{\,0}}$ → special limits at ∞

$$\lim_{x\to\infty} \frac{2}{3} = \boxed{\frac{2}{3}}$$

ln 4.5. $\overset{\text{"DSP"}}{=}$ "$\frac{2\infty^2+3\infty+1}{3\infty^2+5\infty-2}$" = "$\frac{\infty}{\infty}$"

$$\lim_{x\to\infty} \frac{4x+3}{6x+5} \overset{\text{"DSP"}}{=} \frac{4\cdot\infty+3}{6\cdot\infty+5} = \text{"}\frac{\infty}{\infty}\text{"}$$

$$\lim_{x\to\infty} \frac{4}{6} = \frac{4}{6} = \boxed{\frac{2}{3}}$$

Exp5) Evaluate $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty} \frac{(x^2+1)^{1/2}}{x}$

$\overset{\text{"DSP"}}{=} \frac{\sqrt{\infty^2+1}}{\infty} = \text{"}\frac{\infty}{\infty}\text{"}$

"going back to square 1"

$$\lim_{x\to\infty} \frac{\frac{1}{2}\cdot(x^2+1)^{-1/2}(2x)}{1} = \lim_{x\to\infty} x\,(x^2+1)^{-1/2} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$$

Sometimes L.R. is not the best way to proceed
Referring back to 4.4 on limits at ∞

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$$ → highest degree of denom is x
divide ALL by x

$$\lim_{x\to\infty} \frac{\frac{1}{x}\cdot\sqrt{x^2+1}}{\frac{1}{x}\cdot x}$$

Recall

$\sqrt{x^2} = |x|$

if $x>0$ $\sqrt{x^2}=x$

if $x<0$ $\sqrt{x^2}=-x$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{x^2+1}{x^2}}}{1} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}}}{1}$$

$\frac{1}{x^2} \to 0$ → Special Limit at ∞

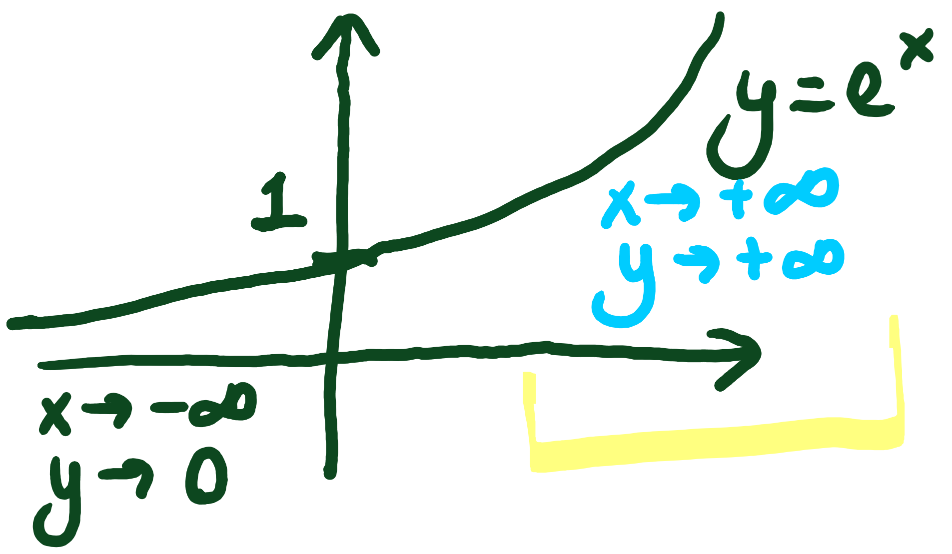
$$= \lim_{x\to\infty} \frac{\sqrt{1}}{1} = \boxed{1}$$

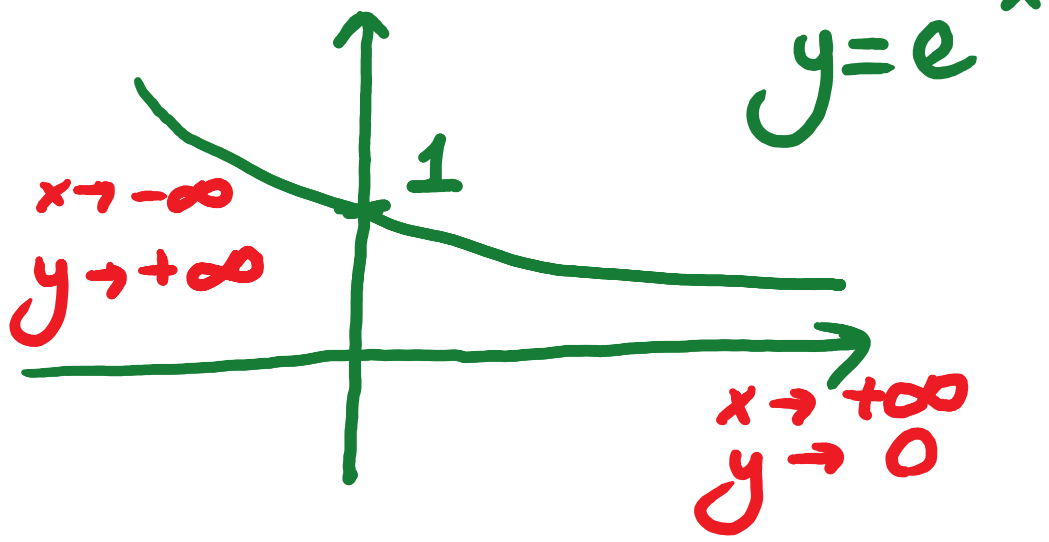
Recall: if $x<0$ $\sqrt{x^2}=-x$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to-\infty} \frac{-\frac{1}{x}\sqrt{x^2+1}}{\frac{1}{x}\cdot x} = \lim_{x\to-\infty} \frac{-\sqrt{\frac{x^2+1}{x^2}}}{1}$$

if $x\to-\infty$; $x<0$

Limits w/ e^x





Exp6) Find <u>all</u> Horizontal Asymptotes of $y=x\cdot e^{-2x}$

$$\lim_{x\to\infty} x\cdot e^{-2x} = \lim_{x\to\infty}\frac{x}{e^{2x}} \overset{\text{"DSP"}}{=} \frac{\infty}{e^{\infty}} = \text{"}\frac{\infty}{\infty}\text{"}$$

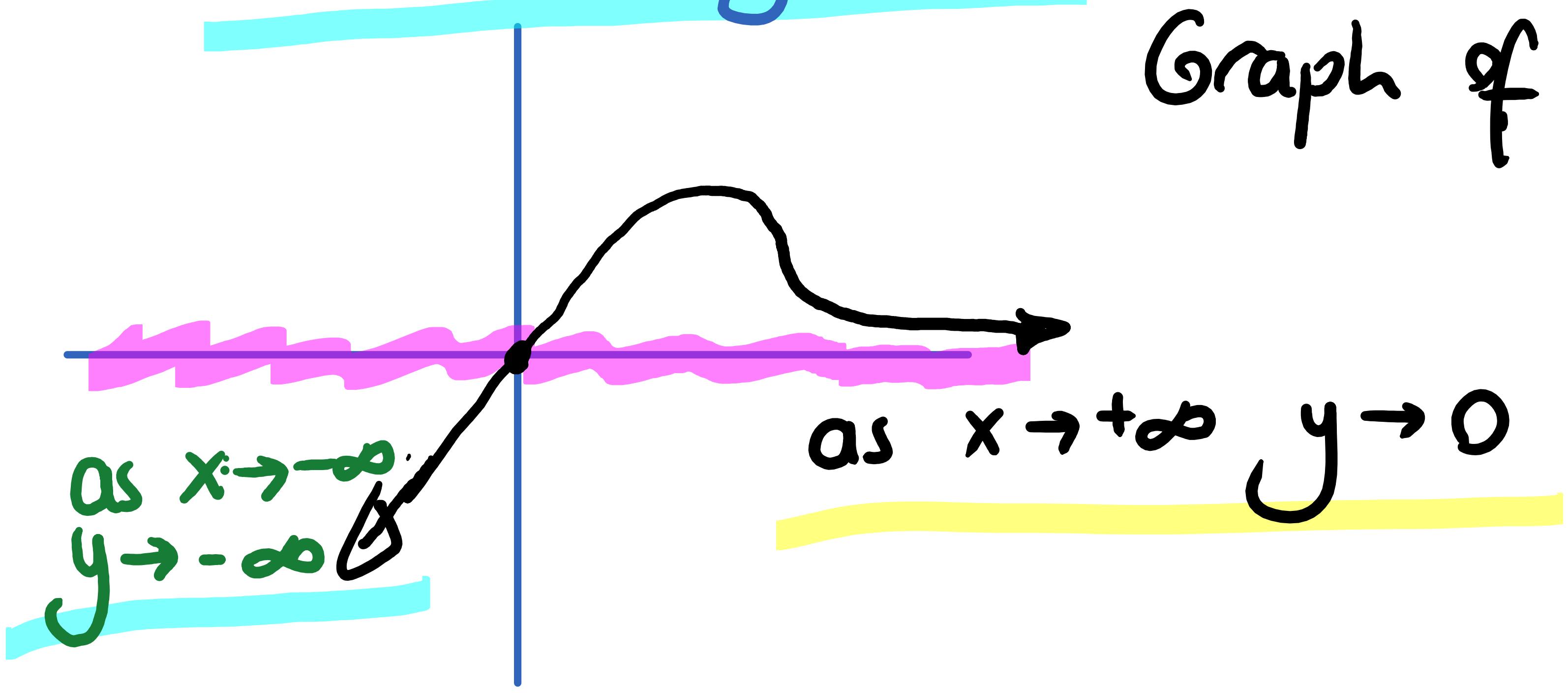
$$\lim_{x\to\infty}\frac{1}{2\cdot e^{2x}} \overset{\text{"DSP"}}{=} \frac{1}{2\cdot e^{2\cdot\infty}} = \frac{1}{2\cdot\infty} = 0$$

as $x\to\infty$ $y\to 0$ (H.A. $y=0$)

$$\lim_{x\to-\infty}\frac{x}{e^{2x}} \overset{\text{"DSP"}}{=} \frac{-\infty}{e^{-2\infty}} = \frac{-\infty}{e^{-\infty}} = \frac{-\infty}{\frac{1}{e^{\infty}}} = -\infty\cdot\infty = -\infty$$

as $x\to-\infty$ $y\to-\infty$ (no H.A)

Graph of $y=x\cdot e^{-2x}$



Other Indeterminate Forms

1^∞, 0^0, ∞^0, $\infty - \infty$, $0 \cdot \infty$ can often be manipulated algebraically into $\frac{0}{0}$, $\frac{\infty}{\infty}$; then eval. by using L.R.

Exp7) Evaluate $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

$$\underline{\underline{\text{"DSP"}}} \left(1 + \frac{1}{\infty}\right)^\infty \stackrel{\text{"DSP"}}{=} 1^\infty$$

let $L = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

$$\boxed{\ln L} = \ln \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

(limit of a log property)

$$\ln L = \lim_{x \to \infty} \left(\ln \left(1 + \frac{1}{x}\right)^x \right)$$

$$= \lim_{x \to \infty} \left(x \cdot \ln \left(1 + \frac{1}{x}\right)\right)$$

Recall $x = \frac{1}{\frac{1}{x}}$

$$\stackrel{\text{"DSP"}}{=} \lim_{x \to \infty} \left(\infty \cdot \ln(1) \right) = \lim_{x \to \infty} (\infty \cdot 0)$$

$$= \lim_{x \to \infty} \left(\frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \right)$$

$$\underline{\underline{\text{"DSP"}}} \frac{\ln\left(1 + \frac{1}{\infty}\right)}{\frac{1}{\infty}} = \text{"}\frac{0}{0}\text{"}$$

Use L.R: $\lim_{x\to\infty}\left(\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}\right) = \lim_{x\to\infty}\left(\frac{\frac{\cancel{-x^{-2}}}{1+\frac{1}{x}}}{\cancel{-x^{-2}}}\right)$

Recall: $[\ln(f(x))]' = \frac{f'(x)}{f(x)}$

$(1+\frac{1}{x})' = (1+x^{-1})'$

$= 0 - x^{-2}$

$= \lim_{x\to\infty}\left(\frac{1}{1+\frac{1}{x}}\right)$

"DSP" $= \frac{1}{1+\frac{1}{\infty}} = \frac{1}{1+0} = 1$

$\ln_e L = 1 \Rightarrow L = e^1 = e$

$\boxed{L = e}$

Exp8) Evaluate $\lim_{x\to(\frac{\pi}{2})^-}\left(x-\frac{\pi}{2}\right)\cdot\tan x$ $\boxed{0\cdot\infty}$

"DSP" $\left(\frac{\pi}{2}-\frac{\pi}{2}\right)\cdot\tan\left(\frac{\pi}{2}\right) = 0\cdot\frac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}} = 0\cdot\frac{1}{0}$

$= 0\cdot\infty$

$= \lim_{x\to(\frac{\pi}{2})^-}\frac{\left(x-\frac{\pi}{2}\right)}{\cot x}$ "DSP" $= \frac{\frac{\pi}{2}-\frac{\pi}{2}}{\cot\frac{\pi}{2}} = $ "$\frac{0}{0}$"

Recall: $\tan x = \frac{1}{\cot x}$

$\cot x = \frac{\cos x}{\sin x}$

$\csc^2 x = \frac{1}{\sin^2 x}$

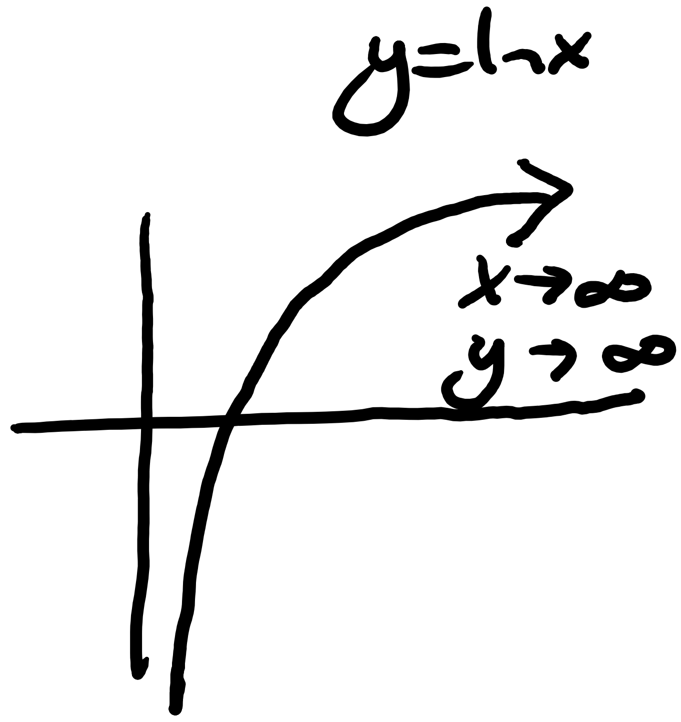
$= \lim_{x\to(\frac{\pi}{2})^-}\frac{1}{-\csc^2 x}$ "DSP" $= \frac{1}{-\frac{1}{\sin^2(\frac{\pi}{2})}}$

$= \boxed{-1}$

Exp 9) Evaluate $\lim_{x\to\infty} x^{\frac{1}{x}}$ $\boxed{\infty^0}$

"DSP" $= \infty^{\frac{1}{\infty}} = \infty^0$

Let $\lim_{x\to\infty} x^{\frac{1}{x}} = L$



$$\ln L = \ln \lim_{x\to\infty} x^{\frac{1}{x}}$$

$$\ln L = \lim_{x\to\infty} \ln x^{\frac{1}{x}}$$

$$\ln L = \lim_{x\to\infty} \left(\frac{1}{x} \cdot \ln x\right)$$

$$\overset{\text{"DSP"}}{=} \frac{\ln \infty}{\infty} = \text{"}\frac{\infty}{\infty}\text{"}$$

$$= \lim_{x\to\infty} \left(\frac{\frac{1}{x}}{1}\right) = \lim_{x\to\infty} \frac{1}{x} \overset{\text{"DSP"}}{=} \frac{1}{\infty} = 0$$

$$\ln_e L = 0$$

$$\boxed{L = e^0 = 1}$$

Exp 10) Evaluate $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$ $\boxed{\infty-\infty}$

$$\text{"DSP"}\ \frac{1}{0}-\frac{1}{\sin 0} = \infty-\infty$$

$$\lim_{x\to 0^+}\left(\frac{1}{x}\frac{(\sin x)}{(\sin x)}-\frac{1}{\sin x}\frac{(x)}{(x)}\right) = \lim_{x\to 0^+}\left(\frac{\sin x - x}{x\cdot\sin x}\right)$$

$$\overset{\text{"DSP"}}{=} \frac{\sin 0 - 0}{0\cdot \sin 0} = \text{"}\frac{0}{0}\text{"}$$

$$\lim_{x\to 0^+}\left(\frac{\cos x - 1}{1\cdot\sin x + x\cdot\cos x}\right) \overset{\text{"DSP"}}{=} \frac{\cos 0 - 1}{\sin 0 + 0\cdot\cos 0} = \text{"}\frac{0}{0}\text{"}$$

$$\lim_{x\to 0^+}\frac{-\sin x}{\cos x + 1\cdot\cos x + x(-\sin x)} \overset{\text{"DSP"}}{=} \frac{-\sin 0}{\cos 0 + \cos 0 + 0} = \frac{-0}{2}$$

$$= \boxed{0}$$