4.6. Optimization in Physics & Life Sciences

Obj: Use math modeling & reasoning and calculus to solve optimization problems.

Def: The process of finding the min or max of a function is called Optimization.

Procedure for Optimization

→ Draw a figure, label all quantities

→ Focus on the quantity to be optimized. (Objective Function). Find a formula to define it.

→ Use conditions (constraints) in the given prob. to eliminate variables in the Obj. Func. The goal is to express the obj. function by using a single variable.

→ Find the practical domain for variables (e.g: dimensions of obj. can NOT be (−) #)

→ Use methods of calculus to obtain the required optimum value.

Exp1) Q8 in Text

The difference of two numbers is 8. Find the smallest possible product

let x and y be the two numbers.

difference $\to (-)$ $\qquad x - y = 8$ (constraint)

product $\to (*)$ $\qquad \underline{x \cdot y}$ $\downarrow$ (min value?) (obj. function)

$$P(x,y) = x \cdot y \quad \text{(obj. f)}$$

Can we eliminate x or y?

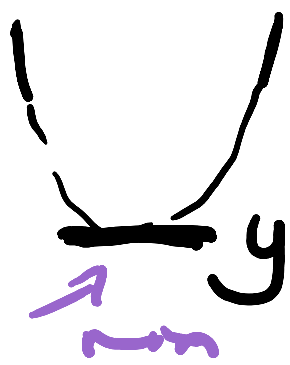
Yes! use the constraint as: $\quad x - y = 8$

$$\begin{array}{r} x - y = 8 \\ +y \quad +y \\ \hline \boxed{x = 8 + y} \end{array}$$

$$P(x,y) = x \cdot y$$

$$\left.P(8+y, y) = (8+y) \cdot y\right\} \quad P(y) = (8+y) \cdot y$$

$$\boxed{\begin{aligned} P(y) &= 8y + y^2 \\ &= y^2 + 8y \end{aligned}}$$



Find $P'(y) = 0$ or DNE

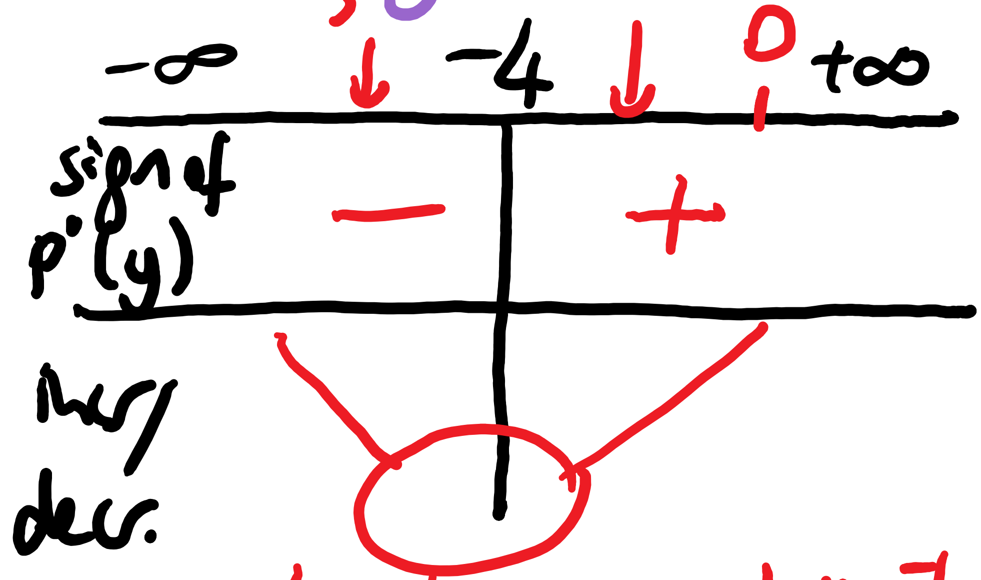
$$P'(y) = (y^2 + 8y)' = \underline{2y + 8 = 0} \quad \text{or DNE}$$

$y = -4$ $\qquad$ none

Construct a sign chart for $P'(y)$ $\boxed{y}$

$\boxed{P'(y) = 8+2y}$ $\quad y=-4$

Domain for y is $(-\infty, +\infty)$



local min at $y=-4$

" global

test P: $y=-5, -3$

$P'(-5) = 8+2(-5) < 0$ $\ominus$

$P'(-3) = 8+2(-3) > 0$ $\oplus$

Global min/max (Sect. 4.1)

We found $y=-4$, use the constraint of $x-y=8$

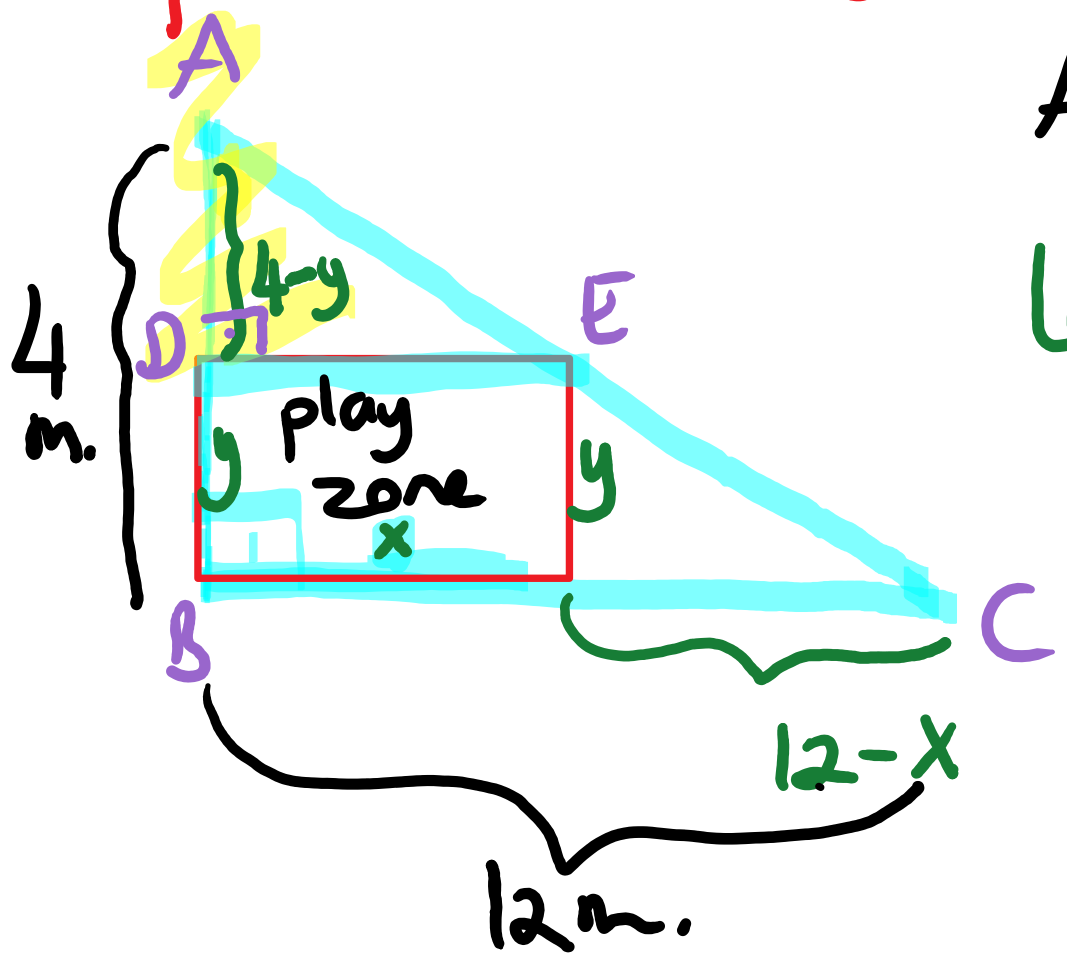
when $\boxed{y=-4} \Rightarrow x-(-4)=8$

$$x+4=8$$
$$\boxed{x=4}$$

The smallest possible product is $4(-4) = \boxed{-16}$.

Exp2) Page 283 in Text

You need to build a rectangular fence to enclose a play zone for children. What's the maximum area for this play zone if it is to fit into a right-triangular plot with sides measuring 4m. and 12m.?



Area of rectangle ↑

let x, y be the length and width of a rect.

Obj. F. → $\underbrace{A(x,y) = x \cdot y}_{\text{max.}}$

constraint → $x = 3(4-y)$

ΔADE and ΔABC share the angle $\hat{A}$.

Both Δ have $\widehat{ADE}$ and $\widehat{ABC}$ as 90°

$\Delta ADE \sim \Delta ABC$ (similarity statement)

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}$$

$$\frac{4-y}{4} = \text{"SKIP"} = \frac{x}{12} \Rightarrow \boxed{3(4-y) = x}$$

constraint

$$\frac{4-y}{\cancel{4}_{1}} = \frac{x}{\cancel{12}_{3}}$$

Re-write the obj. f:

$A(x,y) = x\cdot y$ Constraint: $x = 3(4-y)$

$A(3(4-y), y) = 3(4-y)\cdot y$

$A(y) = 3y(4-y)$ ↑ max } $A(y) = 12y - y^2 \rightarrow \max$

Domain: $0 \leq y \leq 4$ 4 is the max height of the bigger Δ

y in $[0,4]$

$0 \leq x \leq 12$ 12 is the max. base of the bigger Δ

x in $[0,12]$

Find $A'(y)$ & construct a sign chart for $A(y)$

$$A'(y) = (3y(4-y))' = (12y - 3y^2)' = 12 - 6y$$

$A'(y) = 0$ or DNE
none

$$A'(y) = 12 - 6y = 0 \Rightarrow \boxed{y = 2}$$

test p: $y=1$, $y=3$

$A'(1) = 12 - 6 > 0$ (+)

$A'(3) = 12 - 18 < 0$ (−)

sign chart for $A'(y)$

y		0	1	2	3	4	
sign of $A'(y)$	X		+		−		X
incr/decr.				local max at $y=2$			

Is the local max actually the global max?

$A(y) = (12-3y) \cdot y$

$y = 2$ (first-order critical #)

domain of y is $[0,4]$

$A(0) = 0$

$A(4) = 0$

$A(2) = (12-3\cdot 2)\cdot 2$

$= (12-6)\cdot 2$

$= 6 \cdot 2 = \boxed{12\,m^2}$

when $y = 2$ m.

$3(4-y) = x$

$3(4-2) = 3 \cdot 2 = 6 = x$

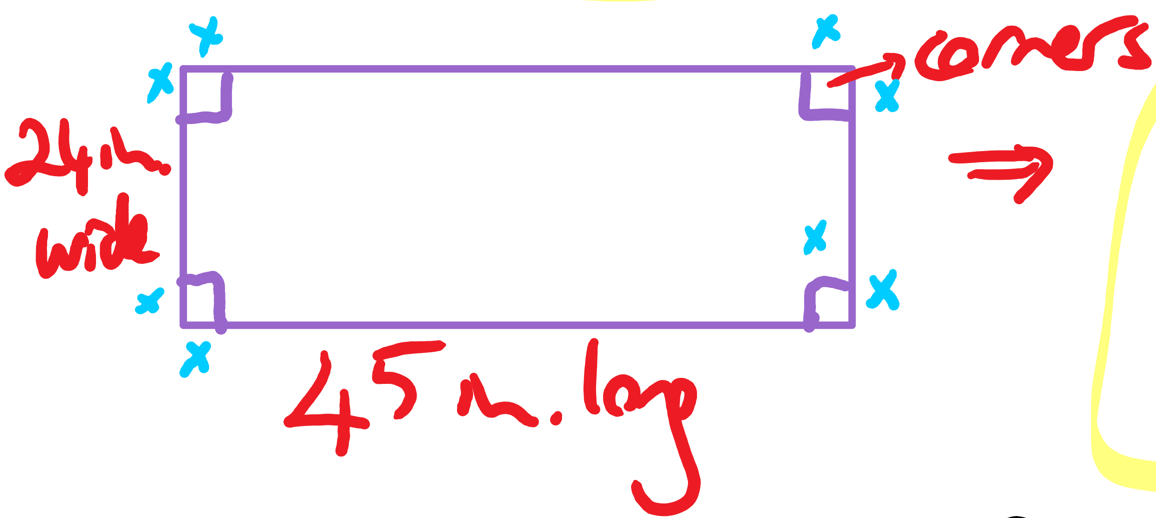
$x = 6$ m.

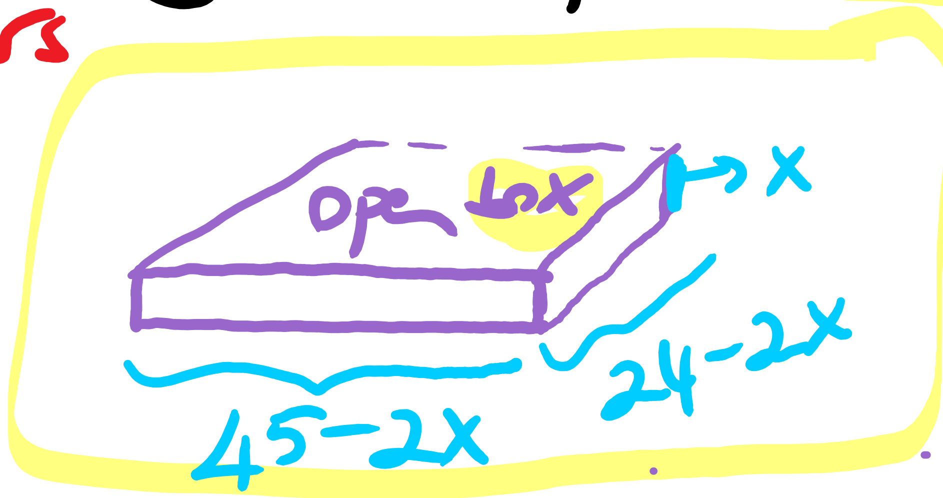
$A(x,y) = 6 \cdot 2 = \boxed{12\,m^2}$

Exp3) Page 284 in Text

Maximizing a Volume

A carpenter wants to make an open-topped box out of a rectangular sheet of tin 24 in. wide and 45 in. long. The carpenter plans to cut congruent squares out of each corner of the sheet and then bend and solder the edges of the sheet upward to form the sides of the box. For what dimensions does the box have the greatest possible volume?





Find dimensions of the box that gives the greast V.

open box

$\ell = 45-2x$

$w = 24-2x$

$h = x$

Constraint

Obj. F. Volume $= \ell \cdot w \cdot h$

$V(x) = (45-2x)(24-2x)\cdot x$

$V'(x) = 0$ or DNE | domain

$$V(x) = (45-2x)(24-2x)\cdot x$$
$$= 4x^3 - 138x^2 + 1080x$$

$$V'(x) = 12x^2 - 276x + 1080$$
$$= 12(x-18)(x-5) = 0$$

$V'(x)$ DNE none

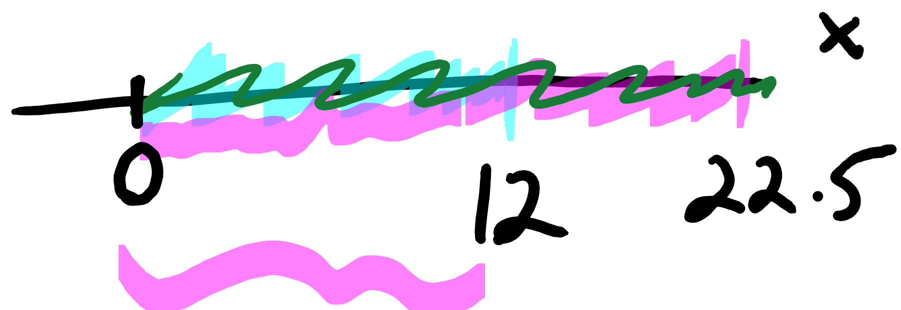
first-order critical #s: $\boxed{\cancel{x=18},\ x=5}$ $[0,12]$

Domain: All dim. of open box should by ≥ 0

$45-2x \geq 0 \Rightarrow 22.5 \geq x \Rightarrow x \leq 22.5$ ①

$24-2x \geq 0 \Rightarrow 12 \geq x \Rightarrow x \leq 12$ ②

$x \geq 0 \Rightarrow x \geq 0$ ③



Domain of x: $[0,12]$

sign chart for $V'(x) = 12(x-18)(x-5)$

0   4   5   6   12

sign of $V'(x)$	X	+	−	X
incr/decr:		↗	↘	

local max at $x=5$

Test P.:
$V'(4) > 0$
$V'(6) < 0$

The box with the largest volume is found when $x = 5$ in. Other dimensions are:

$$l = 45 - 2x$$
$$l = 45 - 10 = 35 \text{ in.}$$

$$w = 24 - 2x$$
$$w = 24 - 2 \cdot 5$$
$$= 14 \text{ in.}$$