4.7. Optimization in Business

Business principles:

Goal#1 Maximize profit

Goal#2 Minimize cost

Goal#1: Profit is maximized when the marginal revenue equals marginal cost.

Review of Terms:

$C(x) \to$ cost function

$p(x) \to$ demand function (price customers pay per item x)

$R(x) = p(x) \cdot x$ (total revenue obtained from the sale of x units)

$\boxed{P(x) = R(x) - C(x)} \to$ total profit function

Goal is to maximize the profit:

$$\boxed{P'(x) = 0}$$

(find first-order critical #s)

Expl) A manufacturer estimates that when x units of a particular item are produced each month, the total cost (in \$) will be:

$$\boxed{C(x)} = \frac{1}{8}x^2 + 4x + 200$$

and all units can be sold at price of $\boxed{p(x) = 49 - x}$ dollars per unit. Determine the price that corresponds to the MAX. PROFIT.

max. profit is when $P(x) = R(x) - C(x)$ is MAX. $(P'(x) = 0)$

$\underbrace{(R'(x) - C'(x) = 0)}$

$$R(x) = p(x) \cdot x = (49 - x) \cdot x$$

$$C(x) = \frac{1}{8}x^2 + 4x + 200$$

$$R'(x) = [49x - x^2]' = 49 - 2x$$

$$C'(x) = \left(\frac{1}{8}x^2 + 4x + 200\right)' = \frac{1}{8} \cdot 2x + 4 = \frac{2x}{8} + 4$$

$R'(x) - C'(x) = 0 \rightarrow$ MAX $P(x)$

$R'(x) = [49x - x^2] = 49 - 2x$

$C'(x) = \left(\frac{1}{8}x^2 + 4x + 200\right)' = \frac{1}{8} \cdot 2x + 4 = \frac{2x}{8} + 4$

$R'(x) - C'(x) = 0 \rightarrow$ MAX $P(x)$

$$49 - 2x - \left(\frac{2x}{8} + 4\right) = 0$$

$$49 - \underset{(8)}{\frac{2x}{1}} - \frac{2x}{8} - 4 = 0$$

$$45 - \frac{16x}{8} - \frac{2x}{8} = 0$$

$$45 - \frac{18x}{8} = 0$$

$$\frac{18x}{8} = \frac{45}{1} \Rightarrow x = \frac{\cancel{45}^{\,5} \cdot \cancel{8}^{\,4}}{\cancel{18}\,\cancel{2}\,1}$$

$\boxed{x = 20 \text{ items}}$

$$\boxed{\begin{array}{l} \text{when } x = 20 \\ p(x) = 49 - 20 \\ \quad = \$29 \\ \text{Final Answer} \end{array}}$$

The price that corresponds to the MAX PROFIT is $29 per unit.

Exp2) "C Level" Q6 on Page 307

Determine the level of production that maximizes profit. number of items (x)

Given: $C(x) = \frac{1}{5}(x+20)$ $\boxed{p(x) = \frac{70-x}{x+30}}$

To MAX. the profit: $P(x) = R(x) - C(x)$

$P'(x) = R'(x) - C'(x)$

$P'(x) = 0$

$$R(x) = x \cdot p(x) = x \cdot \left(\frac{70-x}{x+30}\right) = \frac{70x - x^2}{x+30}$$

$$R'(x) = \frac{(70-2x)(x+30) - (70x - x^2) \cdot 1}{(x+30)^2}$$

$$R'(x) = \frac{\cancel{70x} + 2100 - 2x^2 - 60x \cancel{- 70x} + x^2}{(x+30)^2}$$

$$R'(x) = \frac{2100 - x^2 - 60x}{(x+30)^2}$$

$$C'(x) = \left(\frac{1}{5}(x+20)\right)' = \frac{1}{5}$$

$$R'(x) = C'(x) \Rightarrow (R'(x) - C'(x) = 0)$$

$$\frac{2100 - x^2 - 60x}{(x+30)^2} = \frac{1}{5} \Rightarrow$$

$$\frac{2100-x^2-60x}{(x+30)^2} = \frac{1}{5}$$

$$5(2100-x^2-60x) = (x+30)^2 \rightarrow \cancel{x^2+30^2}$$

$$10500-5x^2-300x = x^2+60x+900$$

$$\begin{array}{r} 900 \\ -10500 \\ \hline -9600 \end{array}$$

$$0 = 6x^2+360x-9600$$
$$= 6(x^2+60x-1600)$$

Quadratic Formula:

$$ax^2+bx+c=0 \qquad x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x^2+60x-1600=0$$

$a=1$
$b=60$
$c=-1600$

$$x_{1,2} = \frac{-60 \pm \sqrt{3600-4\cdot1\cdot(-1600)}}{2\cdot1}$$

$$x_{1,2} = \frac{-60 \pm \sqrt{3600+6400}}{2}$$

$$x_{1,2} = \frac{-60 \pm \sqrt{10000}}{2}$$

$$x_{1,2} = \frac{-60 \pm 100}{2}$$

$$\cancel{x_1 = \frac{-60-100}{2}} \qquad x_2 = \frac{-60+100}{2}$$

$$x_2 = \frac{40}{2} = \boxed{20}$$

The # of items that max. the profit is 20.

Goal #2 MINIMIZE THE COST

Average cost is MINIMUM at the level of production where the marginal cost equals the average cost ($MC = AC$)

Average cost $A(x) = \boxed{\dfrac{C(x)}{x}}$ $\dfrac{(\text{Total cost})}{\#\text{ of items}}$

MIN

$$A'(x) = \left(\frac{C(x)}{x}\right)' \Rightarrow A'(x) = \frac{C'(x)\cdot x - C(x)\cdot 1}{x^2}$$

"div. by x"

$$A'(x) = \frac{\frac{C'(x)\cdot x}{x} - \frac{C(x)}{x}}{\frac{x^2}{x}}$$

$$A'(x) = \frac{C'(x) - \boxed{\frac{C(x)}{x}}}{x}$$

$$A'(x) = \frac{\boxed{C'(x) - A(x)}}{x} = 0$$

To min. ave. cost

$$\boxed{\begin{gathered} A'(x) = 0 \\ C'(x) = A(x) \end{gathered}}$$

To min. ave. cost

$$\boxed{\begin{array}{l} A'(x)=0 \\ C'(x)=A(x) \end{array}}$$

Exp) Given: $C(x)=\frac{1}{8}x^2+4x+200$, $p(x)=49-x$

Determine the level of production (find the # of item (x)) at which the average cost is minimized.

$$A(x)=\frac{C(x)}{x}=\frac{\frac{1}{8}x^2+4x+200}{x}=\frac{1}{8}x+4+\frac{200}{x}$$

$$C'(x)=\frac{1}{8}\cdot 2x+4$$

A(x) is MIN. when $C'(x)=A(x)$

$$\frac{2x}{8}+\cancel{4}=\frac{x}{8}+\cancel{4}+\frac{200}{x}$$

$$-\frac{x}{8} \qquad -\frac{x}{8}$$

$$\frac{x}{8}=\frac{200}{x} \Rightarrow x^2=1600$$

$$=4^2\times10^2$$

$$\boxed{x=40}$$

The min. average cost occurs when $x=40$ items.

Exp) "B level" Q18 on Page 308

Suppose a manufacturer estimates that, when the market price of a certain product is p, the number of units sold will be $x = -6 \cdot \ln\left(\frac{p}{40}\right)$

It's also estimated that the cost of producing these x units will be:

$$C(x) = 4xe^{-x/6} + 30$$

a) Find the average cost, the marginal cost, and the marginal revenue.

b) Find x that corresponds to MAX profit.

Solution:

a) $A(x) = \frac{C(x)}{x} = \frac{4 \cdot x \cdot e^{\frac{-x}{6}} + 30}{x} = 4 \cdot e^{\frac{-x}{6}} + \frac{30}{x}$

$$mc = c'(x) = (4xe^{-x/6} + 30)' = 4 \cdot e^{-x/6} + 4x \cdot \left(\frac{-1}{6}\right) \cdot e^{-x/6}$$

$$mc = 4 \cdot e^{-x/6}\left(1 - \frac{x}{6}\right)$$

$$mR = R'(x) \Rightarrow R(x) = p(x) \cdot x$$

Re-write $x = -6 \cdot \ln\left(\frac{p}{40}\right)$ to obtain p

$$\frac{x}{-6} = \ln\left(\frac{p}{40}\right)$$

$$e^{-x/6} = \frac{p}{40} \Rightarrow p = 40 \cdot e^{-x/6} \quad (p(x))$$

Recall that $R(x) = p(x) \cdot x$

$$= 40 \cdot e^{-x/6} \cdot x$$

$$MR = R'(x) = \left(40x \cdot e^{-x/6}\right)'$$

$$= 40 \cdot e^{-x/6} + 40x \cdot \frac{-1}{6} \cdot e^{-x/6}$$

$$= 40 \cdot e^{-x/6}\left(1 - \frac{x}{6}\right)$$

b) What level of production of x corresponds to maximum profit?

Profit is maximized when $MR - MC = 0$

$$MR - MC = 0$$

$$40 \cdot e^{-\frac{x}{6}}\left(1 - \frac{x}{6}\right) - 4 \cdot e^{-\frac{x}{6}}\left(1 - \frac{x}{6}\right) = 0$$

$$\underbrace{36 \cdot e^{-\frac{x}{6}}}_{\text{never equals zero}} \underbrace{\left(1 - \frac{x}{6}\right)}_{0} = 0$$

$$1 - \frac{x}{6} = 0 \Rightarrow x = 6$$

How do we know $x=6$ actually provides the MAX PROFIT?

sign chart for $p'(x) = 36 \cdot e^{-\frac{x}{6}} \underbrace{\left(1 - \frac{x}{6}\right)}$

x (# of items) can't be negative

5 7

	0		6	
sign of $p'(x)$		$+$		$-$
incr/decr.				

local/global max at $x=6$

$x=6$ is a first-order critical #

$p'(7) < 0$

$p'(5) > 0$

Maximum profit achieved when 6 units of items are sold.