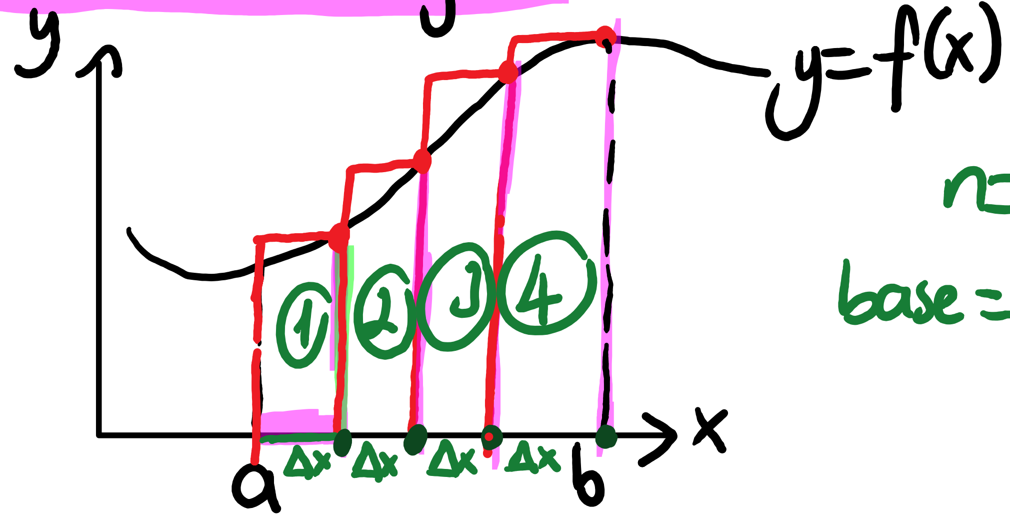
5.2 – 5.3 Area as the limit of a sum
Riemann Sum & Definite Integrals

Goal: To approximate the area under the graph of a continuous function f on $[a,b]$ and above the x-axis by using the sum of areas of rectangles.

To exactly compute the same area by using the limit of a Riemann Sum.

Procedure for Right Endpoint Sum of Areas of Rectangles (Right Riemann Sum)

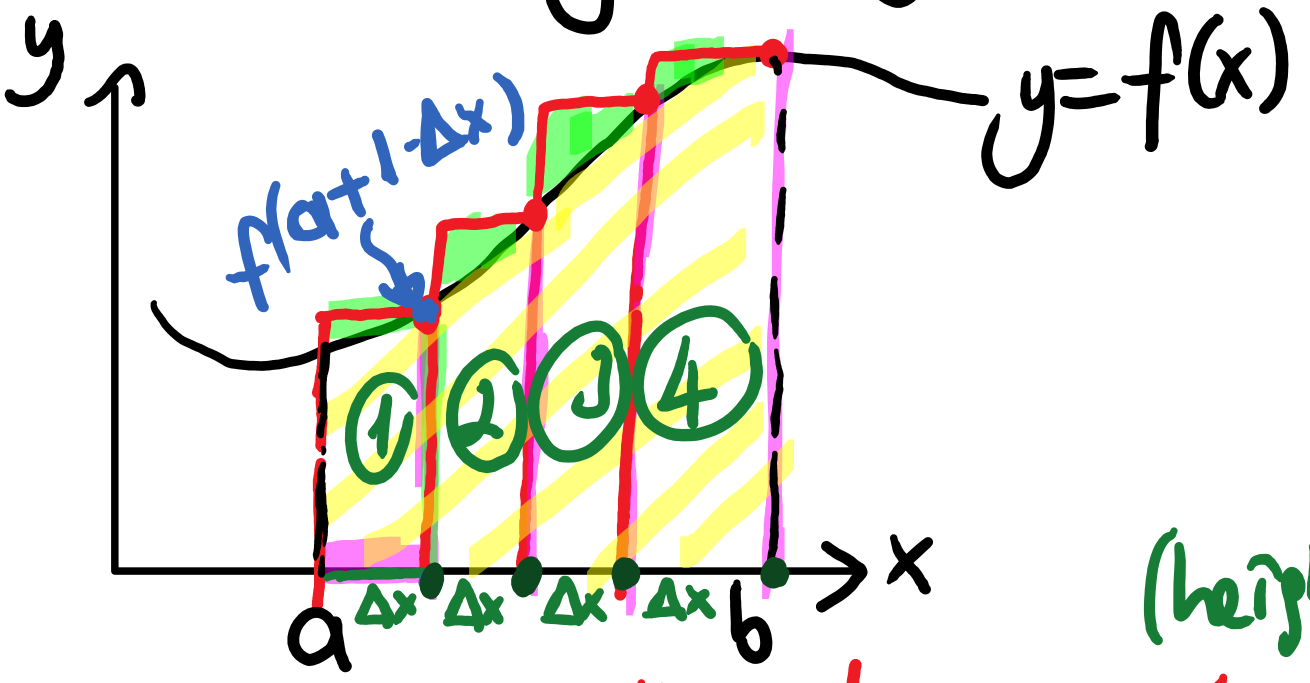
Step 1) Divide $[a,b]$ into n-equal length subintervals. These subintervals on the x-axis determine the base of each rectangle (with different heights)



$n=4$

base $= \boxed{\Delta x = \dfrac{b-a}{n}}$ → total length; n → # of rectangles

Step 2) Determine the right endpoint of each rectangle (x-coordinate) and the height of each rectangle (y-coordinate)



Rectangle #	base	x-coord. of Right E.P.	(height) y-coord. of Left E.P.	base · height Area
R1	Δx	$a+1\cdot\Delta x$	$f(a+1\cdot\Delta x)$	$\Delta x\cdot f(a+1\cdot\Delta x)$
R2	Δx	$a+2\cdot\Delta x$	$f(a+2\cdot\Delta x)$	$\Delta x\cdot f(a+2\cdot\Delta x)$
R3	Δx	$a+3\cdot\Delta x$	$f(a+3\cdot\Delta x)$	$\Delta x\cdot f(a+3\cdot\Delta x)$
R4	Δx	$a+4\cdot\Delta x$	$f(a+4\cdot\Delta x)$	$\Delta x\cdot f(a+4\cdot\Delta x)$

$$S_4 = \Delta x\cdot\left(f(a+1\cdot\Delta x)+f(a+2\cdot\Delta x)+f(a+3\cdot\Delta x)+f(a+4\cdot\Delta x)\right)$$

↳ sum of the 4 rectangles

As n increases (Δx decreases) the approximation improves.

This process is called $\boxed{\text{Riemann Sum}}$ using right endpoints with n rectangles, in a subinterval $\underbrace{[x_{k-1}, x_k]}$

- Right Riemann Sum: start at x_k, $f(x_k)$
- Left Riemann Sum: start at x_{k-1}, $f(x_{k-1})$
- Midpoint " " : $\frac{x_{k-1}+x_k}{2}$, $f\left(\frac{x_{k-1}+x_k}{2}\right)$

Only the Right Riemann sum procedure will be assessed on the FINAL.

Exp1) Estimate the area under the graph of $y=f(x)=x^2$ and above the x-axis on $[0,1]$ with $n=5$ rectangles.

Step1) $\text{base} = \Delta x = \dfrac{b-a}{n}$ $[a,b] \to [0,1]$

$n=5$

$$\Delta x = \frac{1-0}{5} = 0.2$$

Step2) Determine the height, area then total area.

Rectangle #	(Δx) base	x-coord. of Right E.P. $a+1\cdot\Delta x$	(height) y-coord. of Left E.P.	$\Delta x \cdot$ height Area
R_1	0.2	$0+1\cdot(0.2)$ 0.2	$f(0.2)$	$0.2\cdot f(0.2)$
R_2	0.2	0.4	$f(0.4)$	$0.2\cdot f(0.4)$
R_3	0.2	0.6	$f(0.6)$	$0.2\cdot f(0.6)$
R_4	0.2	0.8	$f(0.8)$	$0.2\cdot f(0.8)$
R_5	0.2	1	$f(1)$	$0.2\cdot f(1)$

$$S_5 = 0.2\left(f(0.2)+f(0.4)+f(0.6)+f(0.8)+f(1)\right)$$

$f(x) = x^2$

$$S_5 = 0.2\left(f(0.2)+f(0.4)+f(0.6)+f(0.8)+f(1)\right)$$

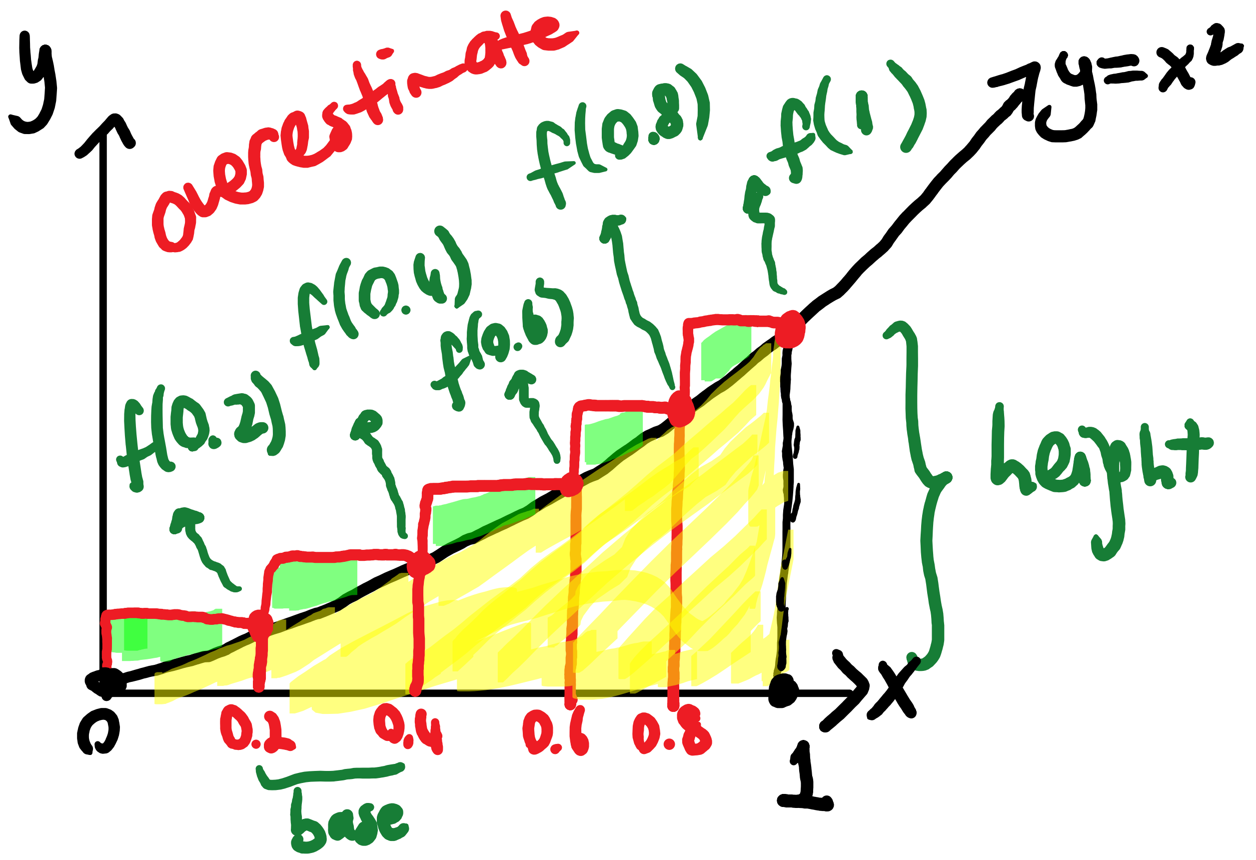
$$= 0.2\left(0.2^2+0.4^2+0.6^2+0.8^2+1^2\right)$$

$$= 0.2\left(\underbrace{0.04+0.16}_{0.20}+\underbrace{0.36+0.64}_{1}+1\right)$$

$$= 0.2(0.2+1+1) = (0.2)\cdot(2.2) = \underline{0.44}$$

$$= \frac{2}{10} \times \frac{22}{10}$$

overestimate



$[0,1] \rightarrow [a,b]$

$y = x^2$

$n = 5$

Summation Notation for Sums

$$\sum_{k=1}^{n} a_k$$

$k \to$ index of summation

$1 \to$ lower limit of sum

$n \to$ upper limit of sum

$\Sigma \to$ uppercase Sigma (summation symbol)

$a_k \to$ general term

Exp) $\displaystyle\sum_{k=3}^{7} a_k = a_3 + a_4 + a_5 + a_6 + a_7$

$= (a_1 + a_2 + \cdots + a_7) - (a_1 + a_2)$

Exp) $\displaystyle\sum_{J=2}^{7} J^3 = 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$

Pg 340 $\to$ summation formulas

(not required to memorize for FINAL)

$$\sum_{k=1}^{n} 1 = n$$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Limit of Riemann Sums

Area under the graph of $y=f(x)$, $f(x) \geq 0$ on $[a,b]$ and the x-axis: ($n \to$ # of rectangles)

$$S_n = \underbrace{\left[f(a+1\cdot\Delta x)+f(a+2\cdot\Delta x)+\dots+f(a+n\cdot\Delta x)\right]}_{\text{height of each rect.}} \underbrace{\Delta x}_{\text{base}}$$

$$= \sum_{k=1}^{n} f(a+k\cdot\Delta x)\cdot\Delta x$$

As $n \to \infty$ $(\Delta x \to 0)$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(a+k\cdot\Delta x)\cdot\Delta x = A$$

$S_n \to$ approximate sum by using Right Riemann sum

$A \to$ exact area under the graph of $f(x)$ on $[a,b]$

$A \to$ is called the integral of f on $[a,b]$

$$\boxed{A = \int_a^b f(x)\cdot dx}$$

$a \to$ lower limit of integration

$b \to$ upper limit of integration

$x \to$ variable of integration

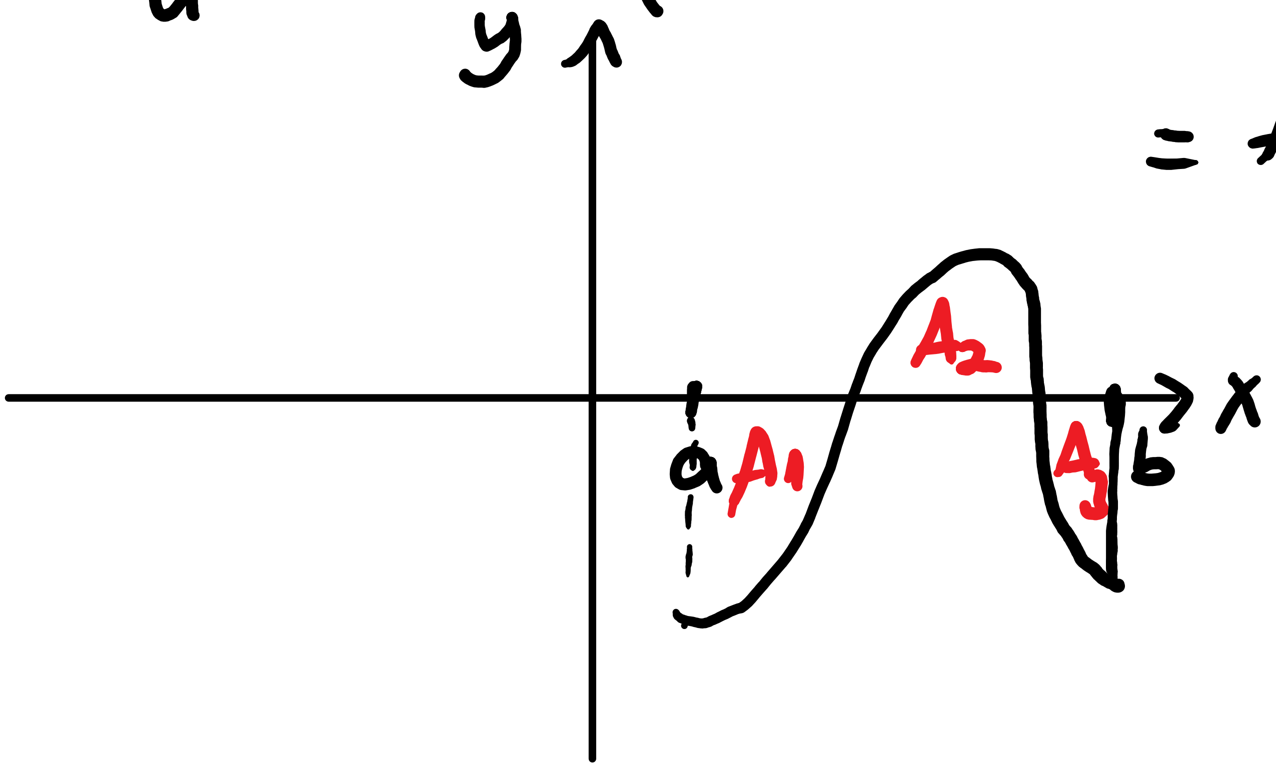
$f(x) \to$ integrand

Properties of Definite Integrals

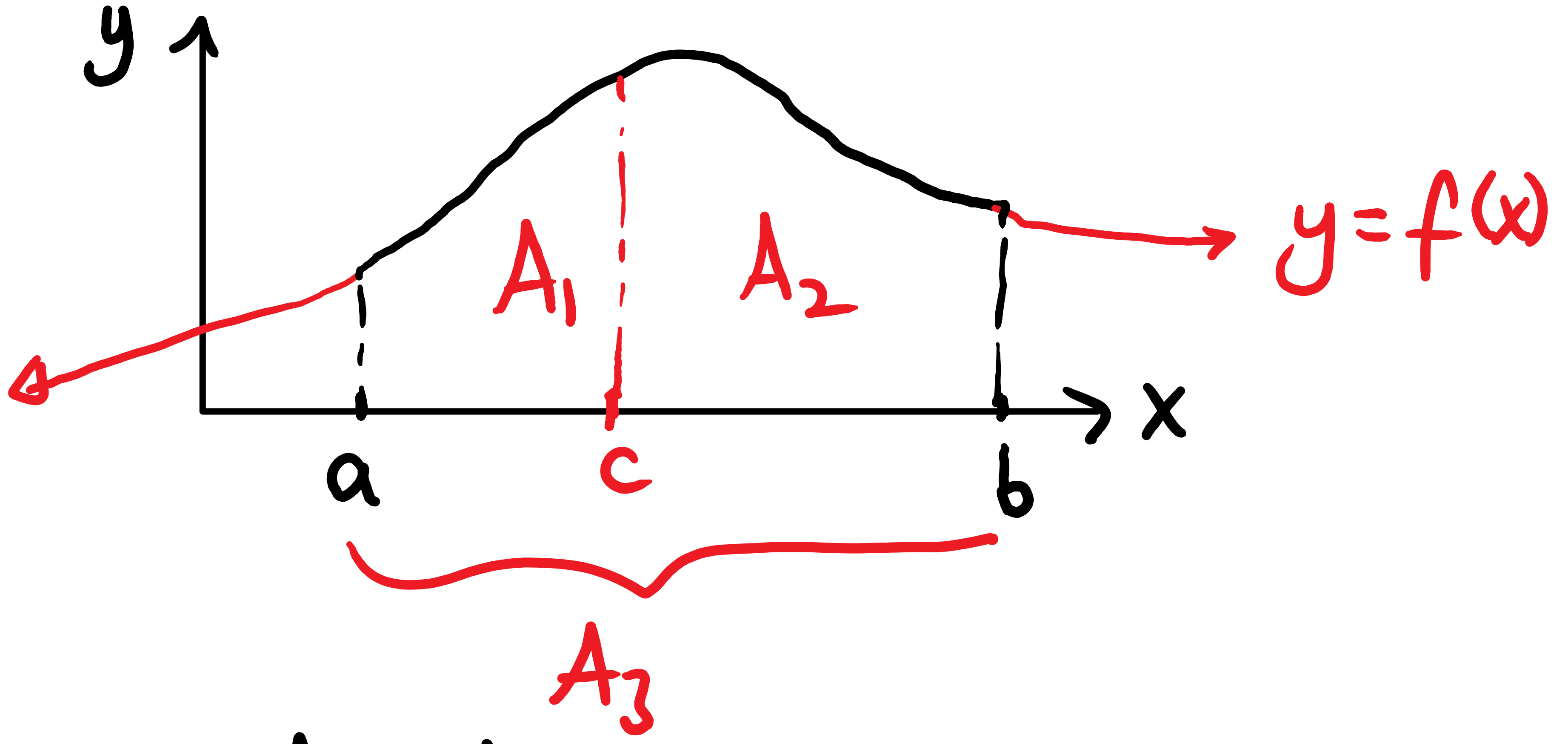
1) $\displaystyle\int_a^a f(x)\,dx = 0$

2) $\displaystyle\int_a^b f(x)\,dx = \begin{pmatrix}\text{area above} \\ \text{the x-axis}\end{pmatrix} - \begin{pmatrix}\text{area below} \\ \text{the x-axis}\end{pmatrix}$

$= A_2 - (A_1 + A_3)$



3) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



$$A_3 = A_1 + A_2$$

Re-visit Exp: $f(x)=x^2$, $[a,b] \to [0,1]$

$$A = \lim_{\substack{n\to\infty \\ (\Delta x \to 0)}} \sum_{k=1}^{n} f(a+k\cdot\Delta x)\cdot \Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(k\cdot\frac{1}{n}\right)\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k^2}{n^2}\right)\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k^2}{n^3}\right)$$

$a=0$

$f(a+k\cdot\Delta x)=$
$f(0+k\cdot\Delta x)$

$\Delta x = \frac{b-a}{n}$

$\Delta x = \frac{1-0}{n}$

$f(x)=x^2$

$f\left(\frac{k}{n}\right) = \frac{k^2}{n^2}$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k^2}{n^3}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty} \left(\frac{n(n+1)(2n+1)}{6n^3}\right)$$

$$= \lim_{n\to\infty} \left(\frac{n(2n^2+n+2n+1)}{6n^3}\right) \leftarrow \text{div ALL by } n^3$$

$$= \lim_{n\to\infty} \left(\frac{n(2n^2+3n+1)}{6n^3}\right)$$

$$= \lim_{n\to\infty} \left(\frac{\frac{2n^3}{n^3} + \cancelto{0}{\frac{3n^2}{n^3}} + \cancelto{0}{\frac{n}{n^2}}}{\frac{6n^3}{n^3}}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2}{6}\right) = \frac{1}{3} = 0.\overline{3} \quad (\text{exact area})$$

In conclusion, 0.44 was an overestimation of the exact area of $0.\overline{3}$.