5.4. Fundamental Theorem of Calculus

First Fundamental Theorem of Calculus

If f is continuous on $[a,b]$ and $F'(x)=f(x)$

Then $$\int_a^b f(x)\,dx = \underbrace{F(b)-F(a)}$$
$$= \underbrace{F(x)\Big|_a^b} = \underbrace{\big[F(x)\big]_a^b}$$

$F(x)$ is an antiderivative of $f(x)$.

Find an antiderivative of $f(x)$, $F(x)$, then evaluate F at the limits of integration a and b.

Exp 1) Find the exact area between $f(x)=x^2$ and the x-axis on $[0,1]$

(Evaluate $\int_0^1 x^2 \cdot dx$.) same question (worded differently)

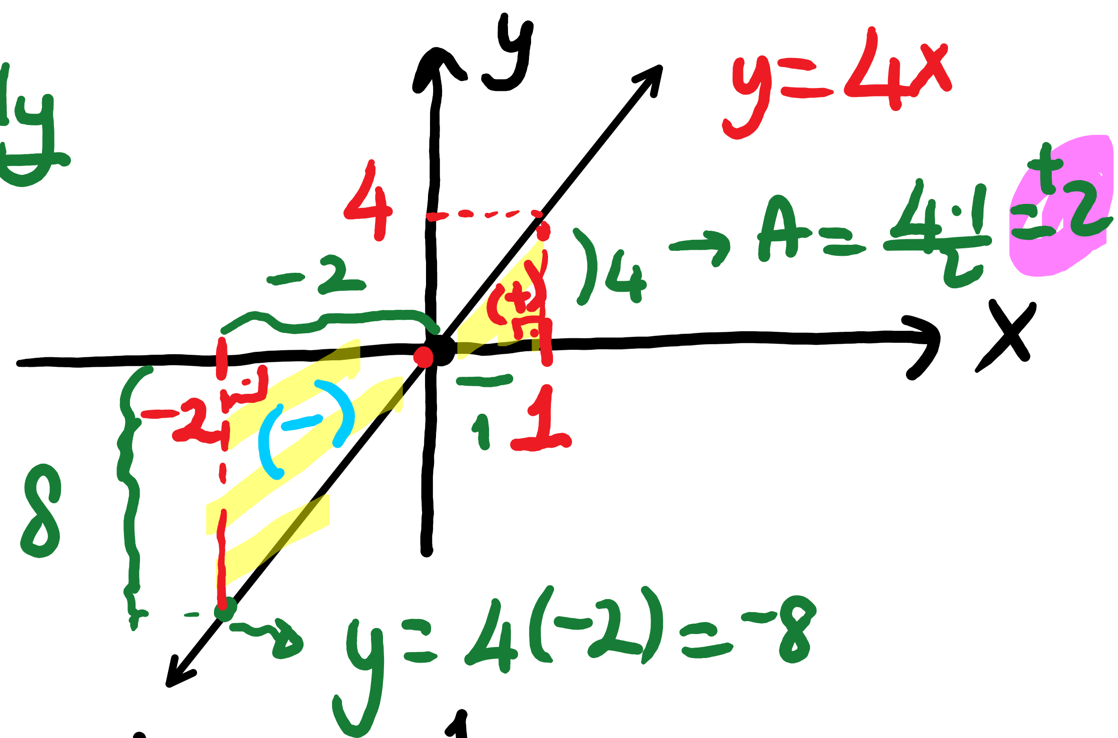
$$\int_0^1 x^2 \cdot dx = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

(Interchanging the limits)

Exp2) Evaluate $\int_{-2}^{1} 4x\cdot dx$ by using the definition of the definite integral and also by using the first fundamental theorem of calculus.

Graphically



$A = \frac{2\cdot 8}{2}$ ←

$A = -8$

Total Area $= \int_{-2}^{1} 4x\cdot dx = 2 - 8 = -6$

By using Calculus

$$\int_{-2}^{1} 4x\cdot dx = 4\cdot\frac{x^2}{2}\Big|_{-2}^{1} = 2x^2\Big|_{-2}^{1} = 2\cdot(1)^2 - 2\cdot(-2)^2$$
$$= 2 - 2\cdot 4 = -6$$

$\int_{-2}^{1} 4u\cdot du$ ↪ $\int^{1} 4t\cdot dt$

x → variable of integration

"dummy variable"

$$\int_{-2}^{1} 4x \cdot dx = 4 \cdot \frac{x^2}{2} \Big|_{-2}^{1} = 2x^2 \Big|_{-2}^{1} = 2 \cdot (1)^2 - 2 \cdot (-2)^2$$

$$= 2 - 2 \cdot 4 = -6$$

$$\int_{-2}^{1} 4u \cdot du$$

$x \to$ variable of integration

$$\hookrightarrow \int_{-2}^{1} 4t \cdot dt$$

"dummy variable"

Exp) Find the area under the curve $y=\cos x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

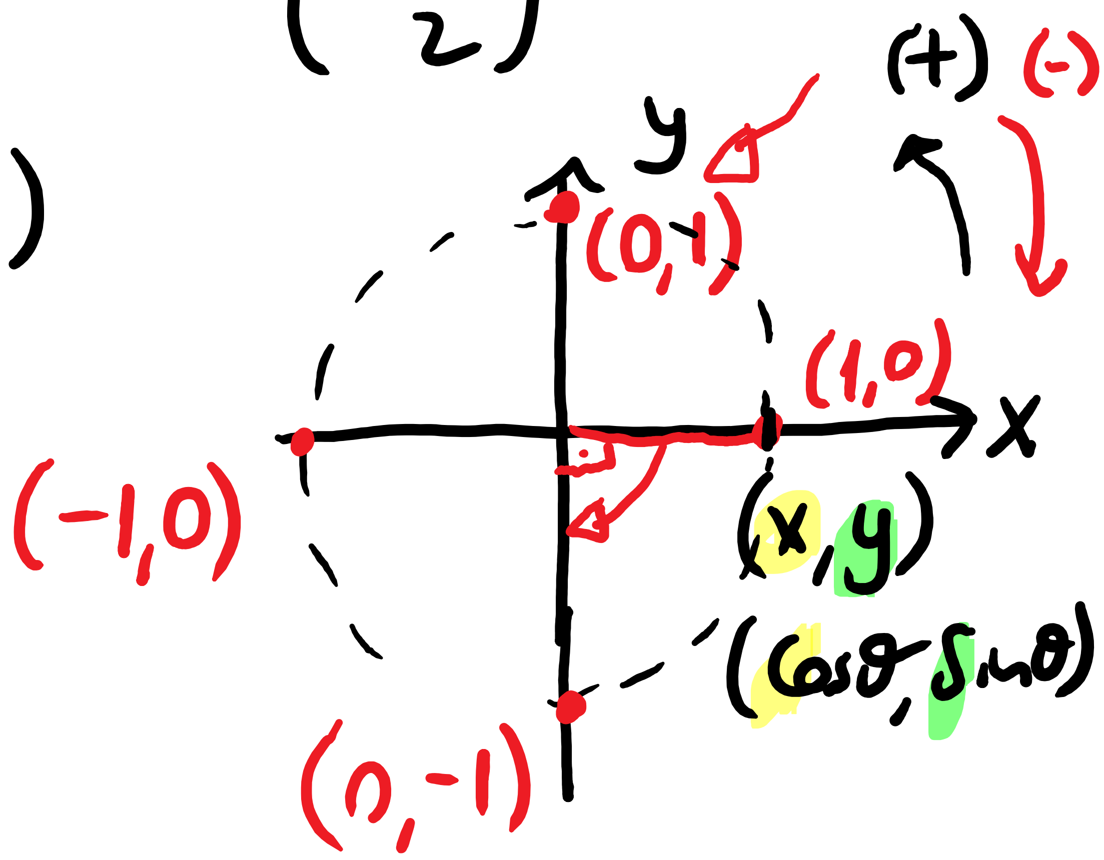
$$\int_{-\pi/2}^{\pi/2} \cos x \cdot dx = \left(\sin x + \cancel{C}\right)\Big|_{-\pi/2}^{\pi/2}$$

$$= \left(\sin\left(\frac{\pi}{2}\right) + \cancel{C}\right) - \left(\sin\left(\frac{-\pi}{2}\right) + \cancel{C}\right)$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{-\pi}{2}\right)$$

$$= 1 - (-1)$$

$$= \boxed{2}$$



Relationship between definite & indefinite integral

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$$

Examples - Evaluate

a) $\int_{-3}^{5} (-10) \cdot dx = (-10x) \Big|_{-3}^{5} = (-10 \cdot 5) - (-10 \cdot -3)$

$= -50 - 30$

$= \boxed{-80}$

b) $\int_{4}^{9} \left(\frac{1}{\sqrt{x}} + x \right) \cdot dx = \int_{4}^{9} (x^{-1/2} + x^{1}) \cdot dx$

$= \left(\frac{x^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)} + \frac{x^2}{2} \right) \Big|_{4}^{9} = \left(\frac{x^{1/2}}{1/2} + \frac{x^2}{2} \right) \Big|_{4}^{9}$

$= \left(2\sqrt{x} + \frac{x^2}{2} \right) \Big|_{4}^{9} = \left(2 \cdot \sqrt{9} + \frac{9^2}{2} \right) - \left(2 \cdot \sqrt{4} + \frac{4^2}{2} \right)$

$= \left(6 + \frac{81}{2} \right) - (4 + 8) = 6 + \frac{81}{2} - 12 = \frac{81}{2} - \frac{6}{1}_{(2)}$

$= \underbrace{\frac{81}{2} - \frac{12}{2}} = \boxed{\frac{69}{2}}$

c) $\int_{-2}^{2} |x| \cdot dx$

$$|x| = \begin{cases} x \geq 0 & x \\ x < 0 & -x \end{cases}$$

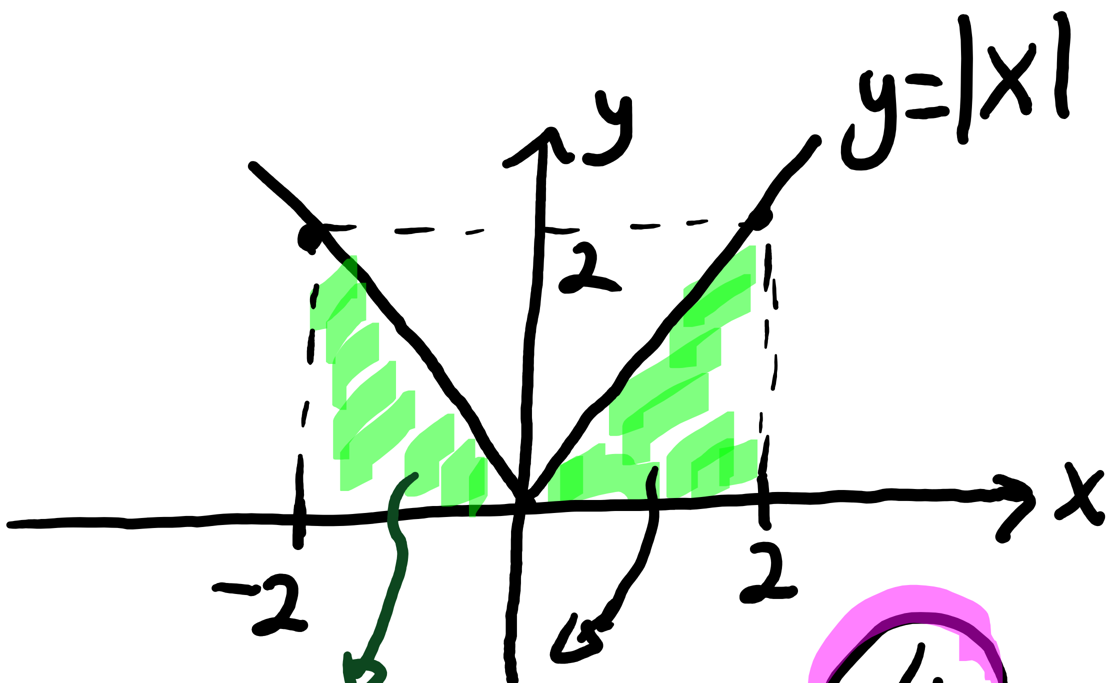
$$= \underbrace{\int_{-2}^{0} |x|dx}_{} + \underbrace{\int_{0}^{2} |x|dx}_{} = \int_{-2}^{0} (-x) \cdot dx + \int_{0}^{2} x \cdot dx$$

$$= -\frac{x^2}{2}\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_{0}^{2} = \left(\frac{-0^2}{2} \underbrace{- \left(\frac{-(-2)^2}{2}\right)}_{-\left(-\frac{4}{2}\right) \; -(-2)}\right) + \left(\frac{2^2}{2} - \frac{0^2}{2}\right)$$

$$= (0 + 2) + (2 - 0)$$

$$= \boxed{4}$$

Graphically: $\int_{-2}^{2} |x| dx = ?$



$A = \frac{2 \cdot 2}{2} = 2$

4

The Second Fundamental Theorem of Calculus

$$F(x) = \int_a^x \underbrace{f(t)}_{} \cdot dt$$

$x \rightarrow$ upper limit of integration is a variable

integrand doesn't contain x.

Theorem: Let $f(t)$ be continuous on $[a,b]$ and G is defined as $G(x) = \int_a^x f(t) \cdot dt$ on $[a,b]$. Then, G is an antiderivative of f on $[a,b]$ that is:

$$G(x) = \int_a^x f(t) \cdot dt \leftarrow$$

$$\frac{dG}{dx} = G'(x) = \frac{d}{dx}\left[\int_a^x f(t) \cdot dt\right] = f(x)$$

on $[a,b]$

Exp) $F(x) = \int_2^x t^2 \cdot dt = \frac{x^3}{3} - \frac{2^3}{3} = \frac{x^3}{3} - \frac{8}{3}$

$$G'(x) = \frac{d}{dx}\left[\int_a^x f(t)\cdot dt\right] = f(x)$$

on $[a,b]$

Exp) Differentiate $F(x) = \int_7^x (2t-3)\cdot dt$

$$F'(x) = \frac{d}{dx}\left[\int_7^x \underbrace{(2t-3)}_{f(t)}\cdot dt\right] = f(x) = 2x-3$$

Exp) Differentiate $G(z) = \int_z^5 \frac{\sin u}{u}\cdot du$

$$G(z) = -\int_5^z \frac{\sin u}{u}\cdot du$$

$$G'(z) = \frac{d}{dz}\left[-\int_5^z \underbrace{\frac{\sin u}{u}}_{f(u)}\cdot du\right] = -\frac{\sin z}{z}$$

Q29 "C Level"

Find the area of the region under the curve of $y = \sec^2 x$ on $\left[0, \frac{\pi}{4}\right]$.

$$\int_0^{\pi/4} \sec^2 x \cdot dx = \tan x \Big|_0^{\pi/4} = \tan\left(\frac{\pi}{4}\right) - \tan 0$$

$$= 1 - 0 = \boxed{1}$$

Q17 "B Level"

Evaluate $\displaystyle\int_1^2 \frac{x^3+1}{x^2} \cdot dx$

$$\int_1^2 \left(\frac{x^3}{x^2} + \frac{1}{x^2}\right) \cdot dx = \int_1^2 \left(x + \frac{1}{x^2}\right) \cdot dx = \int_1^2 \left(x + x^{-2}\right) dx$$

$$= \left(\frac{x^2}{2} + \frac{x^{-1}}{-1}\right)\Big|_1^2 = \left(\frac{2^2}{2} + \frac{2^{-1}}{-1}\right) - \left(\frac{1^2}{2} + \frac{1^{-1}}{-1}\right)$$

$$= \left(2 - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right) = \underbrace{2 - \frac{1}{2} - \frac{1}{2}}_{-1} + 1 = 2$$

Q51 – "Level B"

Evaluate $\int_0^2 f(x) \cdot dx$ where $f(x) = \begin{cases} x^3, & \text{if } 0 < x < 1 \\ x^4, & \text{if } 1 \leq x \leq 2 \end{cases}$

$$\int_0^1 x^3 \cdot dx + \int_1^2 x^4 \cdot dx = \frac{x^4}{4}\Big|_0^1 + \frac{x^5}{5}\Big|_1^2$$

$$= \left(\frac{1^4}{4} - \frac{0^4}{4}\right) + \left(\frac{2^5}{5} - \frac{1^5}{5}\right)$$

$$= \frac{1}{4} - 0 + \underbrace{\frac{32}{5} - \frac{1}{5}}_{} = \frac{1}{4}_{(5)} + \frac{31}{5}_{(4)} = \frac{5 + 124}{20} = \boxed{\frac{129}{20}}$$

$$= \frac{129}{20} = 6\,\frac{9}{20}_{(5)} = 6\,\frac{45}{100} = \boxed{6.45}$$

$$\begin{array}{r} 6 \\ 20\overline{)129} \\ -120 \\ \hline 9 \end{array}$$