Midterm #2 Review

3.6 Implicit Differentiation

Q36 - B Level

Find an equation of the normal line to the curve

$$x^2 \cdot \sqrt{y-2} = y^2 - 3x - 5 \qquad \underbrace{P(1,3)}_{(x,\,y)}$$

$$2x \cdot \sqrt{y-2} + x^2 \cdot \frac{1}{2}(y-2)^{-\frac{1}{2}} \cdot \frac{dy}{dx} = 2y \cdot \frac{dy}{dx} - 3$$

$$2 \cdot 1\sqrt{3-2} + 1^2 \cdot \frac{1}{2}(3-2)^{-1/2} \cdot \frac{dy}{dx} = 2 \cdot 3 \cdot \frac{dy}{dx} - 3$$

$$2 \cdot 1 + \frac{1}{2} \cdot 1 \cdot \frac{dy}{dx} = 6 \cdot \frac{dy}{dx} - 3$$

$$2 + 3 = \frac{6}{\frac{1}{(2)}} \frac{dy}{dx} - \frac{1}{2} \cdot \frac{dy}{dx}$$

$$\frac{2}{11} \cdot 5 = \frac{11}{2} \cdot \frac{dy}{dx} \cdot \frac{2}{11}$$

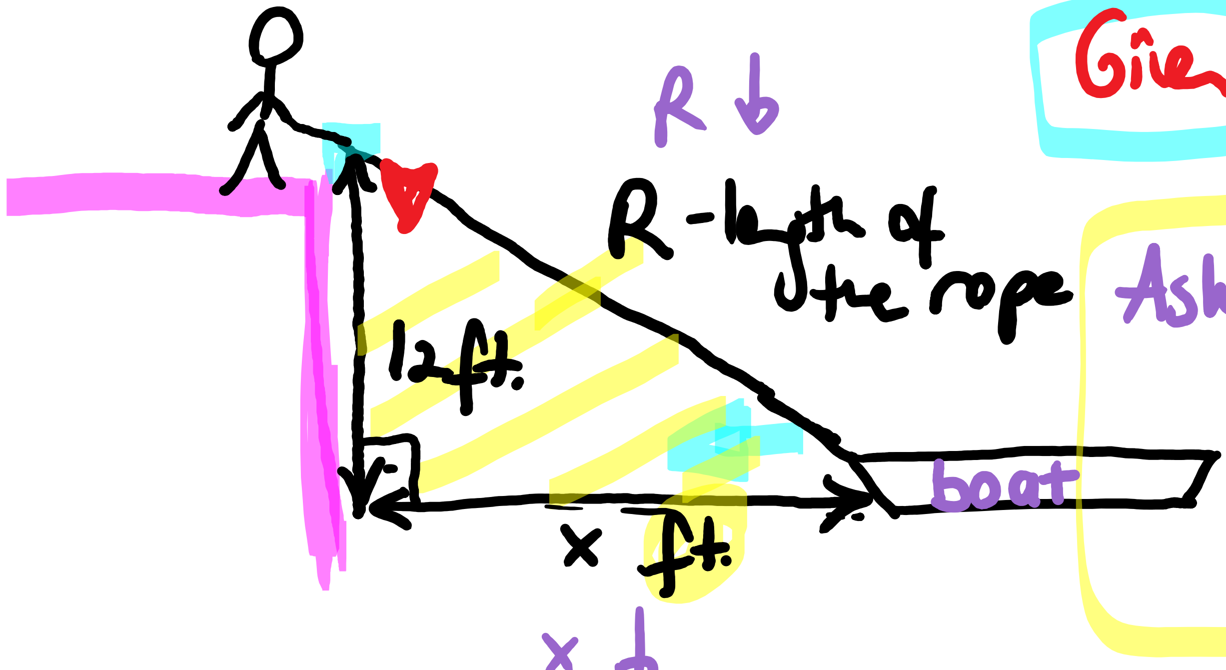
$$\frac{10}{11} = \frac{dy}{dx} \quad (m_{tan})$$

$$m_{normal} = \frac{-1}{m_{tan}} = \frac{-1}{\frac{10}{11}} = \underbrace{\frac{-11}{10}}_{} \qquad \underbrace{(1,3)}_{}$$

Eq. of the normal line: $\underbrace{y - 3 = \frac{-11}{10}(x-1)}$

3.7. Related Rates

Q40 - B level A person is standing at the end of a pier 12 ft. above the water and is pulling in a rope attached to a rowboat at the waterline at the rate of 6 ft. of rope per minute. How fast is the boat moving in the water when it's 16 ft. from the pier?



Given: $\frac{dR}{dt} = -6 \frac{ft}{min.}$

Asked: $\frac{dx}{dt} = ?$ when $x = 16$ ft. specific case

$$R^2 = 12^2 + x^2$$

$$2R \cdot \frac{dR}{dt} = 0 + 2x \cdot \frac{dx}{dt}$$

Use substitution to solve for $\frac{dx}{dt}$.

$$\cancel{2} \cdot \cancel{20}^{5} \cdot (\cancel{-6}^{-3}) = \cancel{2} \cdot \cancel{16}_{\cancel{4}\ 2} \cdot \frac{dx}{dt}$$

$$\boxed{\frac{dx}{dt} = \frac{-15}{2} \frac{ft}{min}}$$

what's R when $x = 16$ ft?

3-4-5 special triangle

The distance between the boat and the pier is decreasing at a rate of 7.5 ft/min

3.8. Linear Approximation

Q44 – B Level

$C(x) = \frac{1}{7}x^2 + 4x + 100$ (cost function)

$p(x) = \frac{1}{4}(80 - x)$ (price per item)

→ Calc. $mc = C'(x)$

a) What's the marginal cost?

b) What's the price (per item) when the marginal cost is 10?

c) Estimate the cost of producing the 11th item.

d) Find the actual cost of producing " " "

a) $mc = C'(x) = \boxed{\frac{2}{7}x + 4}$

b) $mc = 10 = \frac{2x}{7} + 4$, $p(x) = ?$

$$\underbrace{10 = \frac{2x}{7} + 4}$$

$$-4 \qquad\qquad -4$$

$$\frac{7}{2} \cdot 6 = \frac{2x}{7} \cdot \frac{7}{2} \Rightarrow \boxed{x = 21}$$

$p(x) = \frac{1}{4}(80 - x)$

$p(21) = \frac{1}{4}(80 - 21)$

$p(21) = \frac{1}{4} \cdot 59 = 14\frac{3}{4} = \boxed{\$14.75}$

c) $MC(10) = \frac{2}{7} \cdot 10 + 4 = \frac{20}{7} + \frac{4}{1}_{(7)} = \frac{48}{7} \approx \boxed{\$6.86}$

d) $C(x) = \frac{1}{7}x^2 + 4x + 100$

Exact cost

$C(11) - C(10) =$

$\frac{1}{7} \cdot 11^2 + 4 \cdot 11 + \cancel{100} - \left(\frac{1}{7} \cdot 10^2 + 4 \cdot 10 + \cancel{100}\right) = \frac{1}{7}(11^2 - 10^2) + 4 \cdot 11 - 4 \cdot 10$

$= \frac{1}{7}(11-10) \cdot (11+10) + 4$

$= \frac{1}{7} \cdot 1 \cdot 21 + 4 = \$7$

by using Calculus $\quad$ \$6.86 → estimation

by using Algebra $\quad$ \$7 → exact

4.4.

Q29 – "A Level"

$$f(x)=\frac{x^3+1}{x^3-8},\quad f'(x)=\frac{-27x^2}{(x^3-8)^2},\quad f''(x)=\frac{108x(x^3+4)}{(x^3-8)^3}$$

Original f.

Domain:

$$x^3-8\neq 0$$
$$x^3\neq 8$$
$$x\neq 2$$

ARN – {2}

$(-\infty,2)\cup(2,\infty)$

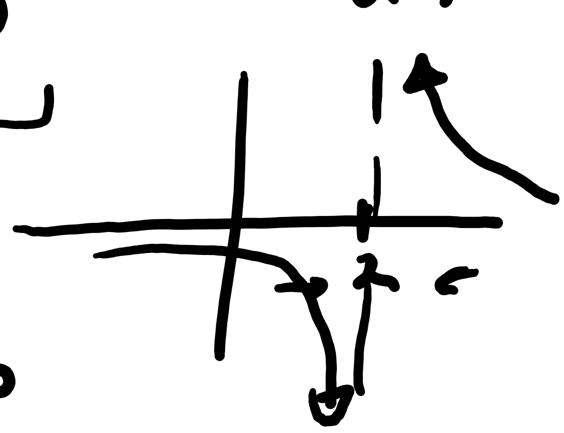
$\cup$ AND

Vertical Asymptote: $\boxed{x=2}$

"OSP"

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}\frac{x^3+1}{\underbrace{x^3-8}}=\frac{2^3+1}{\underset{(-)\ \#}{\text{very tiny}}}=\underbrace{-\infty}$$

V.A.



"OSP"

$$\lim_{x\to 2^+}\frac{x^3+1}{x^3-8}=\frac{2^3+1}{\underset{(+)\ \#}{\text{very tiny}}}=\frac{(+)}{(+)\ \text{very tiny}\ \#}=+\infty$$

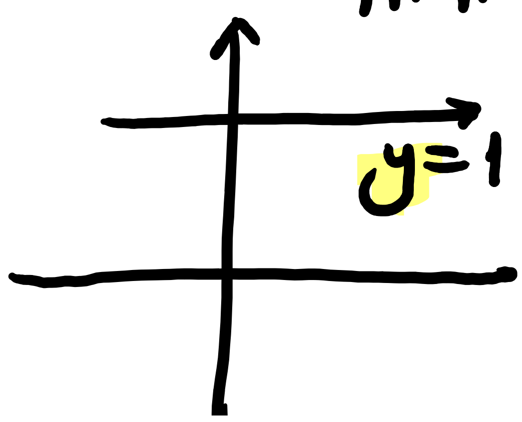
Horizontal Asymptote

$$\lim_{x\to+\infty}\frac{x^3+1}{x^3-8}=\lim_{x\to+\infty}\frac{\frac{x^3}{x^3}+\frac{1}{x^3}}{\frac{x^3}{x^3}-\frac{8}{x^3}}=\lim_{x\to+\infty}\frac{1}{1}=1$$

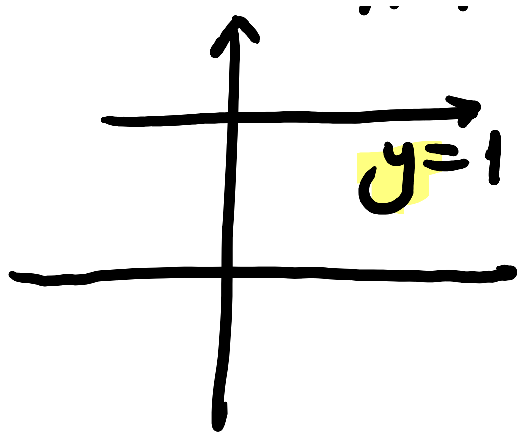
$\boxed{y=1}$

$$\lim_{x\to-\infty}\frac{1+\frac{1}{x^3}}{1-\frac{8}{x^3}}=\lim_{x\to-\infty}f(x)=1$$

H.A.



$$\lim_{x\to-\infty} \frac{\quad x^3}{1-\frac{8}{x^3}} = \lim_{x\to-\infty} f(x) = 1$$



$$f'(x) = \frac{-27x^2}{(x^3-8)^2} \Rightarrow f'(x) = 0 \text{ or DNE}$$

x-coordinates of V.A. in sign chart

First-order critical #s:

$$f'(x) = 0 = \frac{-27x^2}{(x^3-8)^2} \Rightarrow -27x^2 = 0$$

$$\boxed{x = 0}$$

$\boxed{x = 2}$ function is not defined at $x = 2$

sign chart for $f'(x)$

	$-\infty$	0	2	$+\infty$
sign of $f'(x)$	$-$	$-$	$-$	
incr/decr.		$f'(x)=0$		

Decreasing on $(-\infty, 2) \cup (2, \infty)$

$$f''(x) = \frac{108x(x^3+4)}{(x^3-8)^3}$$

Include $x=2$ in the 2nd der. sign chart

$f''(x)=0$ or DNE

2nd order crit #s

$x=0$, $x=-1.58$

$x=\sqrt[3]{-4}$

$f''(x) = 108x\cdot(x^3+4)=0$

$x=0$, $x^3+4=0$

$$\boxed{f''(x) = \frac{108x(x^3+4)}{(x^3-8)^3}}$$

$x^3=-4$

$x=\sqrt[3]{-4} \approx -1.58$

Points of Infl.

sign chart for $f''(x)$

	$-\infty$	-10	$\sqrt[3]{-4}$	-1	0	1	2	5
sign of $f''(x)$		$-$		$+$		$-$		$+$
concave up/down		down		up		down		up
			PoI		PoI		V.A. $x=2$	

$f(0) = -\frac{1}{8}$

$f(\sqrt[3]{-4}) = \frac{1}{4}$

Test points:

$$f''(-10) = \frac{-\,(-10^3+4)}{-} = \frac{(-)\cdot(-)}{(-)} = (-)$$

$$f''(-1) = \frac{-\,(-1^3+4)}{(-1^3-8)^3} = \frac{(-)\cdot(+)}{(-)} = (+)$$

$$f''(1) = \frac{+\,(1^3+4)}{(1^3-8)^3} = \frac{(+)\cdot(+)}{(-)} = (-)$$

concave down on $(-\infty, \sqrt[3]{-4}) \cup (0, 2)$

up on $(\sqrt[3]{-4}, 0) \cup (2, +\infty)$

4.5.
16) Evaluate the limit, justify the use of "L.R"

$$\lim_{x\to 0} \frac{x^2+\sin(x^2)}{x^2+x^3} \overset{\text{"DSP"}}{=} \frac{0+\sin 0}{0+0} = \text{"}\frac{0}{0}\text{"}$$

L.R.

$$\lim_{x\to 0} \frac{2x+\cos(x^2)\cdot 2x}{2x+3x^2} \overset{\text{"DSP"}}{=} \frac{0+1\cdot 0}{0+0} = \text{"}\frac{0}{0}\text{"}$$

L.R.

$$\lim_{x\to 0} \frac{2+2x\cdot(-\sin(x^2))\cdot 2x+\cos(x^2)\cdot 2}{2+6x} \overset{\text{"DSP"}}{=} \frac{2+1\cdot 2}{2+0}$$

0

$$= \frac{4}{2} = \boxed{2}$$

4.7. Q18 - "B Level"

Suppose a manufacturer estimates that, when the market price of a certain product is p, the number of units sold will be: $x = -6 \cdot \ln\left(\frac{p}{40}\right)$

It's also estimated that the cost of producing these x units will be:

$$C(x) = 4x \cdot e^{-x/6} + 30$$

a) Find the average cost, the marginal cost, and the marginal revenue.

b) What level of production x corresponds to maximum profit?

a) Average cost $(A(x)) = \frac{C(x)}{x} = \frac{4x \cdot e^{-x/6} + 30}{x}$

$$A(x) = 4 \cdot e^{-x/6} + \frac{30}{x}$$

marginal cost $(MC = C'(x)) = (4x \cdot e^{-x/6} + 30)'$

$$= 4 \cdot e^{-x/6} + 4x \cdot \left(-\frac{1}{6}\right) \cdot e^{-x/6} + 0$$

$$= 4e^{-x/6}\left(1 - \frac{x}{6}\right)$$

marginal revenue $(MR = R'(x)) = R'(x) = (p(x)\cdot x)'$

$R(x) = p(x)\cdot x$

$x = -6\cdot \ln\left(\frac{p}{40}\right)$

solve for p to obtain $p(x)$:

$\frac{x}{-6} = \ln\left(\frac{p}{40}\right) \Rightarrow e^{-x/6} = \frac{p}{40}$ e

$\Rightarrow p = 40\cdot e^{-x/6}$

$R(x) = p(x)\cdot x = 40\, e^{-x/6}\cdot x = 40x\cdot e^{-x/6}$

$R'(x) = MR = (40x\cdot e^{-x/6})' = 40\cdot e^{-x/6} + 40x\cdot \frac{-1}{6}\cdot e^{-x/6}$

$= 40\cdot e^{-x/6}\left(1-\frac{x}{6}\right)$

b) To maximize profit: $P(x) = R(x) - C(x)$

$\Rightarrow P'(x) = R'(x) - C'(x) = 0$

$P'(x) = 40\cdot e^{-x/6}\left(1-\frac{x}{6}\right) - 4\cdot e^{-x/6}\left(1-\frac{x}{6}\right)$

$= 36\cdot e^{-x/6}\left(1-\frac{x}{6}\right) = 0$

never equals 0 $\quad 1-\frac{x}{6} = 0 \Rightarrow x = 6$

Max. profit when 6 items are produced