5.1. Antidifferentiation

Goal: To study a process called antidifferentiation which reverses differentiation, similar to division reverses multiplication.

Ch. 3, 4 : $f(x) \to f'(x)$

Ch. 5 : $f'(x) \to f(x)$

Antiderivative: A function F is called an antiderivative of a given function f on an interval I if:

$$\boxed{F'(x) = f(x)} \text{ for all } x \text{ in } I.$$

E.g.: Given $F(x) = x^3$ find $F'(x)$.

$$F'(x) = 3x^2$$

What if we know $f(x) = 3x^2$, and the question is "what is the function that has a derivative of $f(x) = 3x^2$?" However;

$F(x) = x^3$ $\qquad$ $F(x) = x^3 + e^5$

$F(x) = x^3 - \frac{22}{5}$

$$F'(x) = 3x^2$$

Theorem: Antiderivatives of a function differ by a constant

If F is an antiderivative of the continuous function f,

Then any other derivative, G, of f must have the form: $G(x) = F(x) + C$

Exp) Find general antiderivatives for:

a) $f(x) = x^5 \Rightarrow F(x) = \dfrac{x^{5+1}}{6} + C$

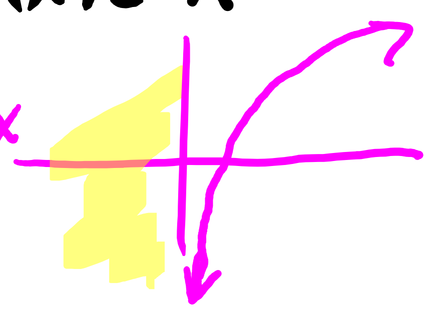
$$F'(x) = \left(\frac{x^6}{6} + C\right)' = \frac{6 \cdot x^5}{6} + 0 = x^5$$

b) $g(x) = x^2 \Rightarrow G(x) = \dfrac{x^3}{3} + C$

c) $h(x) = \dfrac{1}{x} \Rightarrow H(x) = \ln|x| + C$

$h(x) = x^{-1} \Rightarrow \cancel{H(x) = \dfrac{x^{-1+1}}{0} = \dfrac{x^0}{0} = \dfrac{1}{0}}$

$y = \ln x$



d) $s(x) = \sin x \Rightarrow S(x) = ?$

$$S(x) = -\cos x + C$$

Antiderivative Notation

Indefinite Integral The notation

$$\int f(x)\cdot dx = F(x) + C$$

where C is an arbitrarily constant that means F is an antiderivative of f.

F is also called the indefinite integral of f.

$\rightarrow F'(x) = f(x)$ $\quad$ $F'(x)$ Leibniz

$$\frac{dF}{dx} = f(x) \qquad \frac{dF}{dx} \text{ Lagrange}$$

$$dF = f(x)\cdot dx$$

$$\int dF \quad \int f(x)\cdot dx$$

$$\rightarrow F(x) + C = \int f(x)\cdot dx$$

F(x) is the integral of f(x)

$$\underbrace{\int}_{\text{integral sign}} dF = \int f(x)\cdot \underbrace{dx}_{\text{variable of integration}} = F(x) + \underbrace{C}_{\text{constant of integration}}$$

* NO SLOPE FIELDS *

Procedural Rules (Differentiation vs. Integration)

Constant:

$$\frac{d}{dx}\left(c \cdot f(x)\right) = c \cdot \frac{d}{dx}(f(x))$$

vs.

$$\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$$

Sum, difference

$$\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$$

vs.

$$\int (f \pm g)\,dx = \int f \cdot dx \pm \int g \cdot dx$$

Power Rule

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

vs

$$\int x^n \cdot dx \begin{cases} \text{if } n \neq -1 & \dfrac{x^{n+1}}{n+1} + C \\ \text{if } n = -1 & \ln|x| + C \end{cases}$$

Q26 - text
"B level"

Find the indefinite integral:

$$\int \frac{x^2+\sqrt{x}+1}{x^2} \cdot dx$$

$$\int \frac{x^2}{x^2} \cdot dx + \int \frac{\sqrt{x}}{x^2} \cdot dx + \int \frac{1}{x^2} \cdot dx$$

$$= \int 1 dx + \int \frac{x^{1/2}}{x^2} \cdot dx + \int (x^{-2}) \cdot dx$$

$$= \int 1 \cdot dx + \int (x^{\frac{1}{2}-2}) \cdot dx + \int (x^{-2}) dx$$

$$= x + \int x^{-\frac{3}{2}} \cdot dx + \int (x^{-2}) \cdot dx$$

$$= x + \frac{x^{-1/2}}{-1/2} + \frac{x^{-1}}{-1} + C$$

$$= x - 2 \cdot x^{-1/2} - x^{-1} + C$$

$$= x - 2 \cdot \frac{1}{\sqrt{x}} - \frac{1}{x} + C$$

Application of Integrals in Business

Review of 4.7: $C(x)$, $R(x)$, $P(x)$, $A(x) = \frac{C(x)}{x}$

$$\downarrow \quad \downarrow \quad \downarrow$$

$C'(x)$, $R'(x)$, $P'(x)$, $A'(x)$

Exp) A manufacturer estimates that the marginal revenue of a certain item is $R'(x) = 240 + 0.1x$ when x units are produced. Find the demand function $p(x)$.

$$R(x) = p(x) \cdot x$$

Given: $R'(x) \rightarrow \underbrace{\int R'(x) \cdot dx = R(x) \rightarrow \frac{R(x)}{x} = p(x)}_{\text{steps to follow}}$

$$R'(x) = 240 + 0.1x$$

$$\int R'(x) \cdot dx = R(x) \Rightarrow \int (240 + 0.1x^1) \cdot dx$$

$$\Rightarrow 240x + 0.1\frac{x^2}{2} + C = R(x)$$

when $x = 0$ $R(0) = 0$ (initial value)

$$240x + 0.1\frac{x^2}{2} + C = R(x)$$

$R(0) = 0$

$$R(0) = 240 \cdot 0 + \frac{0.1 \cdot (0)^2}{2} + C = 0$$

$$C = 0$$

$$R(x) = 240x + \frac{0.1x^2}{2}$$

$$p(x) = \frac{R(x)}{x} = \frac{240x + \frac{0.1x^2}{2}}{x} = \underbrace{240 + 0.05x}_{\text{demand fuction}}$$

Application of Integrals in Physics

Particle motion:

$s(t) \rightarrow$ position function

$v(t) \rightarrow$ velocity function

$a(t) \rightarrow$ acceleration function

$t \rightarrow$ time

$$\frac{ds}{dt} = v(t)$$

$$ds = v(t) \cdot dt$$

$$\int ds = \int v(t) \cdot dt$$

$$s(t) = \int v(t) \cdot dt$$

$$\frac{dv}{dt} = a(t) \text{ (Differentiation)}$$

(Integral)

$$dv = a(t) \cdot dt$$

$$\int dv = \int a(t) \cdot dt$$

$$v(t) = \int a(t) \cdot dt$$

Exp) MathXL Prb.

A particle moves on a coordinate line w/ acceleration $\frac{d^2s}{dt^2} = 45\sqrt{t} - \frac{24}{\sqrt{t}}$, subject to the conditions that $\overbrace{\frac{ds}{dt}}^{v(t)} = 8$ and $s = 17$ when $t = 1$. Find the velocity and position in terms of t.

Given: $a(t)$

conditions

Asked: $v(t)$, $s(t)$

$$a(t) = 45\sqrt{t} - \frac{24}{\sqrt{t}} \qquad \left(\frac{dv}{dt} = a(t),\ v(t) = \int a(t)\,dt\right)$$

$$v(t) = \int \left(45\sqrt{t} - \frac{24}{\sqrt{t}}\right) \cdot dt = \int (45 \cdot t^{1/2} - 24 \cdot t^{-1/2})\,dt$$

$$= 45 \cdot \frac{t^{3/2}}{3/2} - 24 \cdot \frac{t^{1/2}}{1/2} + C_1$$

Given at $t=1$, $v(t)=8$

$$v(1) = \overset{15}{\cancel{45}} \cdot 1^{3/2} \cdot \frac{2}{\cancel{3}} - 24 \cdot 1^{1/2} \cdot \frac{2}{1} + C_1 = 8$$

$$= \underbrace{30 - 48}_{} + C_1 = 8$$

$$\Rightarrow \quad -18 + C_1 = 8$$

$$C_1 = 26$$

$$v(t) = 45 \cdot \frac{t^{3/2}}{3/2} - 24 \cdot \frac{t^{1/2}}{1/2} + 26$$

$$v(t) = 30 \cdot t^{3/2} - 48 \cdot t^{1/2} + 26$$

$$v(t) = 30 \cdot t^{1} \cdot t^{1/2} - 48 \cdot t^{1/2} + 26$$

$$v(t) = 30 t \cdot \sqrt{t} - 48 \cdot \sqrt{t} + 26$$

Find $s(t)$ $\quad s(t) = \int v(t) \cdot dt$

$$s(t) = \int (30 \cdot t^{3/2} - 48 \cdot t^{1/2} + 26) \cdot dt$$

$$= 30 \cdot \frac{t^{5/2}}{5/2} - 48 \cdot \frac{t^{3/2}}{3/2} + 26 \cdot t^{1} + C_2$$

when $t=1$, $s(1) = 17$

$$s(1) = \cancel{30}^{6} \cdot \frac{2}{\cancel{5}} \cdot 1^{5/2} - \cancel{48}^{16} \cdot \frac{2}{\cancel{3}} \cdot 1^{3/2} + 26 \cdot 1 + C_2$$

$$17 = \underline{12 - 32} + 26 + C_2$$

$$= -20 + 26 + C_2$$

$$17 = 6 + C_2 \Rightarrow C_2 = 11$$

$$s(t) = 12 t^{5/2} - 32 \cdot t^{3/2} + 26 t + 11$$

$$s(t) = 12 \sqrt{t^5} - 32 \cdot \sqrt{t^3} + 26 t + 11$$

$$= 12 \cdot t^2 \sqrt{t} - 32 t \cdot \sqrt{t} + 26 t + 11$$

Exp) Evaluate the indefinite integral

$$\int (x^2-3)^2 \cdot dx = \int (x^4 - 6x^2 + 9) \cdot dx$$

$$= \frac{x^5}{5} - 6 \cdot \frac{x^3}{3} + 9 \cdot x + C$$

$$= \frac{x^5}{5} - 2x^3 + 9x + C$$