5.5 Integration By Substitution

Goal: To reverse the process of chain rule by using substitution for indefinite and definite integral.

Exp:

$$\int \overbrace{(x^2+3x+5)}^{u} \cdot \overbrace{(2x+3) \cdot dx}^{du} = \int u \cdot du$$

$$(x^2+3x+5)' = \frac{d}{dx}(x^2+3x+5) = (2x+3)$$

Let $\boxed{x^2+3x+5 = u}$

$(2x+3) \cdot dx = du$

Theorem: Integration by Substitution

Let f, g and u be differentiable functions of x such that:

$$f(x) = g(u) \cdot \frac{du}{dx} \Rightarrow f(x) \cdot dx = g(u) \cdot \frac{du}{dx} \cdot dx$$

$$\int f(x) \cdot dx = \int g(u) \cdot du = G(u) + C$$

G is an antiderivative of g.

Exp) Find $\int \underbrace{(2x+7)}_{\text{inside function}}^5 \cdot dx$

Let $2x+7 = u$

$$\frac{\cancel{2}\cdot dx}{\cancel{2}} = \frac{du}{2}$$

$$dx = \frac{du}{2}$$

$$\int u^5 \cdot \frac{du}{2} = \frac{1}{2}\int u^5 \cdot du$$

$$= \frac{1}{2}\cdot\frac{u^6}{6} + C$$

$$= \boxed{\frac{(2x+7)^6}{12} + C}$$

Exp) Find $\int (\underbrace{4-2\cdot\cos\theta}_{\text{inside function}})^3 \cdot \sin\theta \cdot d\theta$

~~Let $\sin\theta = u$~~

~~$\cos\theta \cdot d\theta = du$~~

~~$d\theta = \frac{du}{\cos\theta}$~~

Let $4-2\cdot\cos\theta = u$

$$0 + 2\cdot(+\sin\theta)\cdot d\theta = du$$

$$\frac{\cancel{2}\cdot\sin\theta\cdot d\theta}{\cancel{2}} = \frac{du}{2}$$

$$\sin\theta\cdot d\theta = \frac{du}{2}$$

$$\int u^3\cdot\frac{du}{2} = \frac{1}{2}\int u^3\cdot du = \frac{1}{2}\cdot\frac{u^4}{4} + C$$

$$= \boxed{\frac{1}{8}\cdot(4-2\cdot\cos\theta)^4 + C}$$

Exp) find $\int x \cdot e^{4-x^2} \cdot dx = \int e^{4-x^2} \cdot x \cdot dx$

Let $4 - x^2 = u$

$\frac{\cancel{-2}x \cdot dx}{\cancel{-2}} = \frac{du}{-2}$

$x \cdot dx = -\frac{du}{2}$

$\int e^u \cdot \left(-\frac{du}{2}\right) = -\frac{1}{2} \int e^u \cdot du$

$= -\frac{1}{2} \cdot e^u + C$

$= \boxed{-\frac{1}{2} \cdot e^{4-x^2} + C}$

Exp) find $\int x \cdot (4x-5)^3 \cdot dx$

inside function

Let $4x - 5 = u$

$\frac{4 \cdot dx}{4} = \frac{du}{4}$

$dx = \frac{du}{4}$

$4x - 5 = u$
$+5 \quad +5$
$4x = u + 5 \Rightarrow x = \frac{u+5}{4}$

$\int x \cdot u^3 \cdot \frac{du}{4}$ (no mix-match re-write x as $\frac{u+5}{4}$)

$= \int \left(\frac{u+5}{4}\right) \cdot u^3 \cdot \frac{du}{4}$

$= \frac{1}{16} \int (u+5) \cdot u^3 \cdot du$

$= \frac{1}{16} \int (u^4 + 5u^3) \cdot du$

$$\Rightarrow \frac{1}{16}\int (u^4+5u^3)\cdot du$$

$4x-5=u$

$$= \frac{1}{16}\left(\frac{u^5}{5}+5\cdot\frac{u^4}{4}\right)+C$$

$$= \frac{1}{16}\left(\frac{(4x-5)^5}{5}+\frac{5(4x-5)^4}{4}\right)+C$$

Exp) Find $\int \tan x \cdot dx$

Recall: $\tan x = \dfrac{\sin x}{\cos x}$

$$= \int \frac{\sin x}{\cos x} \cdot dx$$

~~Let's try: $\sin x = u$~~

~~$\cos x \cdot dx = du$~~

~~we have $\dfrac{dx}{\cos x}$ not $\cos x \cdot dx$~~

Let's try again!

$\cos x = u$

$$\frac{-\sin x \cdot dx}{-1} = \frac{du}{-1}$$

$\sin x \cdot dx = -du$

Recall: $\int \frac{dx}{x} = \ln|x| + C$

$$\int \frac{-du}{u} = \int -du \cdot u^{-1} = \cancel{-\frac{u^{-1+1}}{0}}$$

$$\int \frac{-du}{u} = -\ln|u| + C = \boxed{-\ln|\cos x| + C}$$

Substitution with Definite Integration

Theorem: If $f(u)$ is a continuous function of u, and $u(x)$ is a differentiable function of x,

Then:

$$\int_a^b f[u(x)]\cdot u'(x)\cdot dx = \int_{u(a)}^{u(b)} f(u)\cdot du$$

Exp) Evaluate $\int_1^2 (4x-5)^3 \cdot dx$

1st method: with changing the limits of integration (keep u)

Let $4x-5=u$

$\frac{4\cdot dx}{4} = \frac{du}{4}$

$dx = \frac{du}{4}$

$\cancel{2} \to$ when $x=2$; $4\cdot 2-5=3=u$ 3

$\int u^3 \cdot \frac{du}{4}$

$\cancel{1} \to$ when $x=1$; $4\cdot 1-5=-1=u$ -1

$$\int_{-1}^{3} u^3\cdot\frac{du}{4} = \frac{1}{4}\int_{-1}^{3} u^3\cdot du = \frac{1}{4}\cdot\frac{u^4}{4}\Big|_{-1}^{3} = \frac{1}{16}\left(3^4-(-1)^4\right)$$

$$= \frac{1}{16}(81-1) = \boxed{5}$$

2nd method: (w/out changing the limits of integration) (re-write w/ the original variable)

Recall: $4x-5=u$

$$\Rightarrow \frac{u^4}{16}\Big|_{-1}^{3} = \frac{(4x-5)^4}{16}\Big|_{1}^{2}$$

$$= \frac{(4\cdot 2-5)^4}{16} - \frac{(4\cdot 1-5)^4}{16}$$

$$= \frac{3^4}{16} - \frac{(-1)^4}{16} = \frac{3^4-1}{16} = \frac{81-1}{16} = \boxed{5}$$

Exp) Evaluate $\int_0^5 \sqrt{3t+1}\cdot dt = \int_0^5 \underbrace{(3t+1)}_{u}^{1/2}\cdot dt$

Let $3t+1=u$

$\frac{\cancel{3}\cdot dt}{\cancel{3}} = \frac{du}{3}$

$dt = \frac{du}{3}$

$$\int u^{1/2}\cdot\frac{du}{3} = \frac{1}{3}\int u^{1/2}\cdot du$$

$$= \frac{1}{3}\cdot\frac{u^{3/2}}{3/2} = \frac{1}{3}\cdot\frac{2}{3}(3t+1)^{3/2}\Big|_0^5$$

* change u back to 3t+1 *

$$= \frac{2}{9}\left[(3\cdot 5+1)^{3/2} - (3\cdot 0+1)^{3/2}\right]$$

$$= \frac{2}{9}\left[16^{3/2} - 1^{3/2}\right] = \frac{2}{9}\left[(4^{\cancel{2}})^{3/\cancel{2}} - 1\right]$$

$$= \frac{2}{9}\left[4^3 - 1\right] = \frac{2}{9}\left[64-1\right] = \frac{2}{\cancel{9}}\cdot \overset{7}{\cancel{63}}$$

$$= \boxed{14}$$

Official List of Prb. — 5.5

Q13) "C Level"

Find $\displaystyle\int \sin(\underbrace{4-x}_{u}) \cdot dx$

Let $4-x=u$

$0 - dx = du$

$\frac{0-dx}{-1} = \frac{du}{-1}$

$dx = -du$

$$\int \sin(u)\cdot(-du) = -\int \sin(u)\,du$$

$$= \cos(u) + C$$

$$= \cos(4-x) + C$$

verify: $\frac{d}{dx}\left[\cos(4-x)+C\right] = -\sin(4-x)\cdot(-1)$

$= +\sin(4-x)$

Q33) "B level"

Evaluate $\displaystyle\int_0^1 \frac{5x^2 \cdot dx}{\underbrace{2x^3+1}}$

Let $2x^3+1 = u$

$\dfrac{\cancel{6}x^2 \cdot dx}{\cancel{6}} = \dfrac{du}{6}$

$x^2 \cdot dx = \dfrac{du}{6}$

$\displaystyle\int \frac{5 \cdot \frac{du}{6}}{u} = \frac{5}{6}\int \frac{du}{u}$

$= \dfrac{5}{6} \ln|u| + C$

$= \dfrac{5}{6} \cdot \ln|2x^3+1| \Big|_0^1$

$= \dfrac{5}{6}\left[\ln(2\cdot1^3+1) - \ln(2\cdot0^3+1)\right]$

$= \dfrac{5}{6}\left[\ln 3 - \cancel{\ln 1}^{\to 0}\right] = \boxed{\dfrac{5}{6} \cdot \ln 3}$

Q27 – "C Level"

Find $\int \frac{\ln x}{x} \cdot dx$

Let $\ln x = u$

$\frac{1}{x} \cdot dx = du$

$\int u \cdot du = \frac{u^2}{2} + C$

$= \underbrace{\frac{(\ln x)^2}{2} + C}$

Recall:

$(\ln x)^2 \neq \ln(x^2)$