Final Review - Part #2

1) Find the absolute minimum and maximum values and points of $f(x) = 1 - x^{2/3}$ on $[-1, 1]$.

The first-order critical #s are:

$f'(x) = 0$ or DNE

$$f'(x) = -\frac{2}{3} \cdot x^{-1/3} = -\frac{2}{3\sqrt[3]{x}}$$

$x = 0$ is a critical #s, endpoints: $x = -1$, $x = 1$

x	$f(x) = 1 - x^{2/3}$
0	$1 - 0^{2/3} = 1$
-1	$1-(-1)^{2/3} = 1 - \sqrt[3]{(-1)^2} = 1 - 1 = 0$
1	$1 - 1^{2/3} = 0$

Absolute maximum value is 1 at $(0, 1)$

Absolute minimum value is 0 at $\underbrace{(-1, 0) \text{ \& } (1, 0)}_{\text{points}}$

(2) Find the derivative of $f(x) = \left(\frac{\sin 5x}{1-2x}\right)^3$

Solution:

$$f'(x) = 3\left(\frac{\sin 5x}{1-2x}\right)^2 \cdot \left(\frac{\sin 5x}{1-2x}\right)'$$

$$= 3\left(\frac{\sin 5x}{1-2x}\right)^2 \cdot \left(\frac{5 \cdot \cos 5x\,(1-2x) - \sin 5x \cdot (-2)}{(1-2x)^2}\right)$$

$$= \frac{3\,(\sin 5x)^2 \cdot [5\cos 5x\,(1-2x) + 2 \cdot \sin 5x]}{(1-2x)^4}$$

(3) Find the equation of the tangent line at $x = 0$ for $y = e^x \cdot \cos x$

Solution:

slope of the tangent line: $y' = e^x \cdot \cos x + e^x \cdot \sin x$

slope of the tangent at $x=0$: $\left.\frac{dy}{dx}\right|_{x=0} = e^0 \cdot \cos 0 + e^0 \cancel{\sin 0}^{\,0}$
$= 1 \cdot 1 = 1$

at $x=0$ $y = e^0 \cdot \cos 0 = 1 \cdot 1 = 1$

Equation of the tangent line:

$y - y_1 = m(x - x_1)$

$y - 1 = 1(x - 0) \Rightarrow y = x + 1$

(4) Find the coordinates of each point on the graph of $h(u) = \frac{1}{\sqrt{u}} \cdot (u+9)$ where the tangent line is horizontal.

Solution:

$h'(u) = 0$ (slope of the tangent line is 0 means horizontal tangent)

$h(u) = u^{-1/2}(u+9) = u^{1/2} + 9u^{-1/2}$

$h'(u) = \frac{1}{2} \cdot u^{-1/2} + 9\left(\frac{-1}{2}\right)u^{-3/2}$

$= \frac{1}{2\sqrt{u}}_{(u)} - \frac{9}{2u\sqrt{u}} = \underbrace{\frac{u-9}{2u\sqrt{u}}}_{\substack{u-9=0 \\ u=9}} = 0$

$u = 9$; $h(9) = \frac{1}{\sqrt{9}}(9+9) = \frac{1}{3} \cdot 18 = 6$

The coordinate is: $(9, 6)$

(5) Evaluate the limit: $\lim_{x \to 0} \frac{\sin^2 x}{2x}$

1st method: $\overset{"1"}{=}$

$\lim_{x \to 0} \frac{\sin x}{2} \cdot \frac{\sin x}{x} \cdot 1 = \lim_{x \to 0} \frac{\sin x}{2} \overset{"DSP"}{=} 0$

2nd method:

$\lim_{x \to 0} \frac{\sin^2 x}{2x} \overset{"DSP"}{=} \frac{0}{0} \overset{L.R}{\Rightarrow} \lim_{x \to 0} \frac{\cancel{2} \cdot \sin x \cdot \cos x}{\cancel{2}}$

$\overset{"DSP"}{=} \sin 0 \cdot \cos 0 = 0$

(6) Evaluate the limit:

$\lim_{x \to 0} \frac{1 - \frac{1}{x+1}}{x}$

1st method:

$\lim_{x \to 0} \frac{1 - \frac{1}{x+1}}{x} = \lim_{x \to 0} \frac{\frac{x+1-1}{x+1}}{x} = \lim_{x \to 0} \frac{1}{x+1} \overset{"DSP"}{=} 1$

2nd method:

$\overset{"DSP"}{=} \frac{1 - \frac{1}{1}}{0} \overset{" "}{=} \frac{0}{0} \overset{L.R}{\Rightarrow} \lim_{x \to 0} \frac{\left(1 - \frac{1}{x+1}\right)'}{x'} = \lim_{x \to 0} \frac{(x+1)^{-2}}{1}$

$\overset{"DSP"}{=} (0+1)^{-2} = 1$

7) Pg 317, Q71

Given: $p(x) \geq 50$ $\qquad$ $p(x) = 150 - x$

$C(x) = 2500 + 30x$

Asked: MAX Profit?

$p(x)$ for MAX profit?

Solution:

$R(x) = p(x) \cdot x = (150 - x) \cdot x = 150x - x^2$

$$P(x) = R(x) - C(x)$$
$$= 150x - x^2 - (2500 + 30x)$$
$$= -x^2 - 2500 + 120x$$

$P'(x) = 0$ (max Profit)

$P'(x) = -2x + 120 = 0 \Rightarrow x = 60$ items

$P(60) = -60^2 - 2500 + 120 \cdot 60 = \boxed{\$1100}$

$p(60) = 150 - 60 = \boxed{\$90 \text{ per item.}}$

8 Page 200, Q54

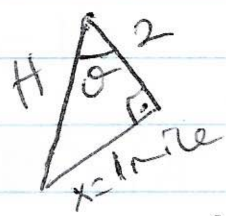
Given: when $x = 1$ mile

$$\frac{d\theta}{dt} = \frac{\pi}{2} \text{ rad/h.}$$

Asked: $\frac{dx}{dt} = ?$

Solution: $\tan\theta = O/A = x/2$

when $x = 1$ mile $\frac{d\theta}{dt} = \frac{\pi}{2}$ rad/h.



$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{2}{\sqrt{5}}} = \frac{\sqrt{5}}{2}$$

$$1^2 + 2^2 = H^2 \Rightarrow H = \sqrt{5}$$

$$\tan\theta = \frac{x}{2} \Rightarrow \sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{2} \cdot \frac{dx}{dt}$$

$$\left(\frac{\sqrt{5}}{2}\right)^2 \cdot \frac{\pi}{2} = \frac{1}{2} \cdot \frac{dx}{dt}$$

$$\frac{5}{4} \cdot \frac{\pi}{2} = \frac{1}{2} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{5\pi}{4} \frac{\text{mile}}{\text{h.}}$$

(9) Page 220, Q75

Use linear approximation to approximate

$$(16.01)^{3/2}+2\sqrt{16.01}$$

Solution:

$$L(x)=f(a)+f'(a)(x-a)$$

$$f(x)=x^{3/2}+2\sqrt{x}\ ,\quad f'(x)=\frac{3}{2}\cdot x^{1/2}+2\cdot\frac{1}{2}\cdot x^{-1/2}$$

$a=16$

$$f(16)=(4^2)^{3/2}+2\sqrt{16}=72$$

$$f'(16)=\frac{3}{2}\cdot(4^2)^{1/2}+(4^2)^{-1/2}=6+\frac{1}{4}=\frac{25}{4}$$

$$L(x)=72+\frac{25}{4}(x-16)$$

$$L(16.01)=72+\frac{25}{4}(16.01-16)=\boxed{72+\frac{1}{6}}$$

(10) Find $\frac{dy}{dx}$ when $x^3 \cdot y^3 + x - y = 1$

Solution:

$$\underbrace{3x^2 \cdot y^3 + x^3 \cdot 3y^2 \cdot \frac{dy}{dx}} + 1 \underbrace{- \frac{dy}{dx}} = 0$$

$$\frac{dy}{dx}\left(3x^3y^2 - 1\right) = -1 - 3x^2y^3$$

$$\boxed{\frac{dy}{dx} = \frac{-1-3x^2y^3}{3x^3y^2-1}}$$

(11) Use the limit definition of derivative to differentiate $f(x) = \frac{1}{2x}$

Solution:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{1}{2(x+\Delta x)}(x) - \frac{1}{2x}(x+\Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{x - (x+\Delta x)}{2x(x+\Delta x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{-\Delta x}{2x(x+\Delta x)}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{2x(x+\Delta x)}$$

$$= \boxed{\frac{-1}{2x^2}}$$

check work! $f'(x) = \left(\frac{1}{2x}\right)' = \left[(2x)^{-1}\right]'$

$$= -(2x)^{-2} \cdot 2$$

$$= \boxed{-1/2x^2}$$

(12) Page 143, Q46

Let $f(x) = \begin{cases} -2x & \text{if } x < 1 \\ \sqrt{x} - 3 & \text{if } x \geq 1 \end{cases}$

Show that $f(x)$ is continuous but not differentiable at $x=1$.

Solution:

Check continuity at $x=1$

$$\lim_{x \to 1^-} (-2x) = -2 \cdot 1 = -2$$

$$\lim_{x \to 1^+} (\sqrt{x} - 3) = \sqrt{1} - 3 = -2$$

$$f(1) = -2$$

All equal therefore, $f(x)$ is continuous at $x=1$

Check differentiability at $x=1$.

$$\lim_{x \to 1^-} \frac{-2x - (\sqrt{1} - 3)}{x-1} = \lim_{x \to 1^-} \frac{-2x+2}{x-1} = -2$$

$$\lim_{x \to 1^+} \frac{\sqrt{x} - 3 - (-2)}{x-1} = \lim_{x \to 1^+} \frac{\sqrt{x} - 1}{x-1} = \lim_{x \to 1^+} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$$

Since $-2 \neq \frac{1}{2}$, $f(x)$ is NOT diff. at $x=1$.