Math 135: Calculus I, Exam #2

Name: Rubric/Answer Key

ID# (last 4 digits): ______ Section: ______

- Problems #1 – #7 are marked as "no partial credit". For these problems, you are not required to show work, and any scratch work will not be considered. You will be awarded none or all of the points, depending only on whether your answer is exactly correct.
- Problems #8 – #11 are marked as "partial credit" For these problems, you are required to show work, and you will be awarded points based on your work. Your work must be written clearly using proper notation. Answers must be justified using techniques that have been taught in this course, and answers without such justification may receive less than full credit – or no credit at all – even if the answer is correct.
- This exam is closed book. Calculators, electronic devices, notes, books, formula sheets, and other outside materials are not allowed. Phones must be turned off and put away.
- Unless otherwise stated, give exact answers: e.g., write π and $\sqrt{2}$ instead of 3.14 and 1.41. However, when an expression simplifies to a well-known value, you must use that value. For example, you must write 1 instead of e^0, and you must write $\frac{1}{2}$ instead of $\cos(\frac{\pi}{3})$.

Problem	Points	Score
1.	5	
2.	5	
3.	5	
4.	5	
5.	5	
6.	5	
7.	5	
8.	10	
9.	10	
10.	10	
11.	10	
12.	25	
Total:	100	

For problems #1 – #7, write your final answer in the appropriate box below.

Problem	Final Answer
1.	-1
2.	-36
3.	$-1/2$
4.	e^6
5.	local max at $x=1$
6.	$y=\frac{-3}{5}$, $y=\frac{3}{5}$
7.	$x=4$

For problems #1 – #7, you are not required to show work, and any work you do write will not be graded. Write your final answer in the table on Page #2 of the exam.

1. Use linear approximation to estimate $\sin\left(\frac{3\pi}{2}+0.02\right)$. Your answer should be an integer or a simplified fraction of integers.

$L(x)=f(a)+f'(a)(x-a)$

$f(x)=\sin x\ ,\ f'(x)=\cos x$

$f(a)=\sin\left(\frac{3\pi}{2}\right)=-1$

$f'\left(\frac{3\pi}{2}\right)=\cos\left(\frac{3\pi}{2}\right)=0$

$L(x)=-1+0\left(x-\frac{3\pi}{2}\right)$

$L\left(\frac{3\pi}{2}+0.02\right)=\boxed{-1}$

2. Evaluate the limit or determine that it does not exist. If the limit is infinite, then your answer should be $+\infty$ or $-\infty$ (as appropriate), instead of "does not exist". $\lim_{x\to 0}\left(\frac{\sin(6x)-6x}{x^3}\right)$.

"DSP" "$\frac{0}{0}$" $\Rightarrow \overset{LR}{=} \lim_{x\to 0}\left(\frac{6\cdot\cos 6x-6}{3x^2}\right) \overset{\text{"DSP"}}{=}$ "$\frac{0}{0}$"

$\overset{LR}{=} \lim_{x\to 0}\left(\frac{6\cdot 6\cdot(-\sin 6x)}{6x}\right)1 = -36$

OR "DSP" $\frac{0}{0} \Rightarrow \overset{LR}{=} \lim_{x\to 0} \frac{-36\cdot 6\cdot\cos 6x}{6} \overset{\text{"DSP"}}{=} \boxed{-36}$

3. Evaluate the limit or determine that it does not exist. If the limit is infinite, then your answer should be $+\infty$ or $-\infty$ (as appropriate), instead of "does not exist". $\lim_{x\to 1^+}\left(\frac{1}{x-1}-\frac{1}{\ln(x)}\right)$.

$\lim_{x\to 1^+} \frac{\ln x-(x-1)}{\ln x(x-1)} \overset{\text{"DSP"}}{=}$ "$\frac{0-0}{0}$" $\Rightarrow \overset{LR}{=} \lim_{x\to 1^+} \frac{\frac{1}{x}-1}{\frac{1}{x}(x-1)+\ln x\cdot 1}$

$=\lim_{x\to 1^+} \frac{\left(\frac{1}{x}-1\right)}{1-\frac{1}{x}+\ln x} \overset{\text{"DSP"}}{=}$ "$\frac{0}{0}$" $\Rightarrow \overset{LR}{=} \lim_{x\to 1^+} \frac{-\frac{1}{x^2}}{+\frac{1}{x^2}+\frac{1}{x}(x)} = \lim_{x\to 1^+} \frac{-\frac{1}{x^2}}{\frac{1+x}{x^2}}$

$=\lim_{x\to 1^+} \frac{-1}{1+x} = \boxed{\frac{-1}{2}}$

4. Evaluate the limit or determine that it does not exist. If the limit is infinite, then your answer should be $+\infty$ or $-\infty$ (as appropriate), instead of "does not exist". $\lim_{x\to\infty}(e^{3x}+4x)^{2/x}$.

$$L=\lim_{x\to\infty}\left(e^{3x}+4x\right)^{\frac{2}{x}} \Rightarrow \ln L=\lim_{x\to\infty}\left(\ln(e^{3x}+4x)^{\frac{2}{x}}\right)$$

$$\Rightarrow \ln L=\lim_{x\to\infty}\left(\frac{2}{x}\cdot\ln(e^{3x}+4x)\right) \overset{\text{"DSP"}}{=} \text{"}\frac{\infty}{\infty}\text{"}$$

$$\Rightarrow \ln L \overset{LR}{=} \lim_{x\to\infty}\left(\frac{2\cdot\frac{3\cdot e^{3x}+4}{e^{3x}+4x}}{1}\right) \overset{\text{"DSP"}}{=} \text{"}\frac{\infty}{\infty}\text{"}$$

$$\overset{LR}{=} \lim_{x\to\infty}\left(2\cdot\frac{9\cdot e^{3x}}{3\cdot e^{3x}+4}\right) \overset{\text{"DSP"}}{=} \text{"}\frac{\infty}{\infty}\text{"}$$

$$\overset{LR}{=} \lim_{x\to\infty}\frac{18\cdot 3\cdot e^{3x}}{3\cdot 3\cdot e^{3x}} \overset{\text{"DSP"}}{=} 6 \Rightarrow \ln L=6$$

$$\boxed{L=e^{6}}$$

5. Find the x-coordinates of all points, if any, where $f(x)=\dfrac{25\ln(x)+5}{x^5}$ has a local extremum, and classify each of these points as either a local minimum or a local maximum.

$$f'(x)=\frac{25\cdot\frac{1}{x}\cdot x^5-(25\ln x+5)\cdot 5x^4}{x^{10}}=\frac{-5x^4\cdot 25\cdot\ln(x)}{x^{10}}=\frac{-125\ln x}{x^6}$$

$f'(x)=0$ or DNE (domain of $f(x)$ is $(0,\infty)$)

$$f'(x)=\frac{-125\cdot\ln x}{x^6}=0 \Rightarrow \ln x=0 \Rightarrow x=e^0=1$$

	0		1	
sign of $f'(x)$		+		—
inc/dec.				

$$\boxed{\begin{array}{l}\text{local max at } x=1\\ \text{local min NONE}\end{array}}$$

6. Find all horizontal asymptotes of $f(x) = \dfrac{3x+8}{\sqrt{25x^2+x+11}}$. Write "NONE" if f has no horizontal asymptotes.

$$\lim_{x\to\infty} \frac{3x+8}{\sqrt{25x^2+x+11}} = \lim_{x\to\infty} \frac{\frac{3x+8}{x}}{\sqrt{\frac{25x^2+x+11}{x^2}}} = \lim_{x\to\infty} \frac{3+0}{\sqrt{25+0}} = \frac{3}{5}$$

$$\lim_{x\to-\infty} \frac{3x+8}{\sqrt{25x^2+x+11}} = \lim_{x\to-\infty} \frac{\frac{3x+8}{x}}{-\sqrt{\frac{25x^2+x+11}{x^2}}} = \frac{-3}{5}$$

$$y=\frac{3}{5},\ y=\frac{-3}{5}$$

7. If x units of a certain product are produced, the total cost is $C(x) = \frac{1}{2}x^3 + 21x + 64$. Find the level of production that minimizes the average cost.

$$A(x) = \frac{C(x)}{x} \Rightarrow A(x) = \frac{1}{2}x^2 + 21 + \frac{64}{x}$$

$A'(x)=0$ when $C'(x) = A(x)$

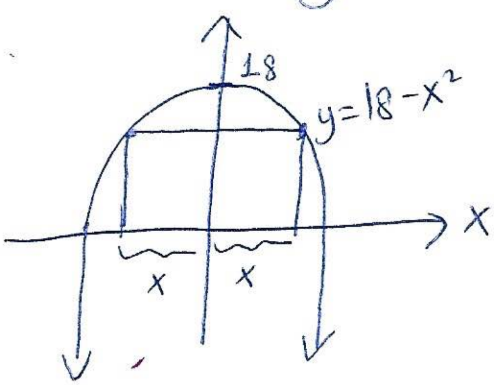
$$A'(x) = x - \frac{64}{x^2} = 0 \Rightarrow \boxed{x=4}$$

For problems #8 – #11, you must show all work, and your work will be graded. Your work should be clear and use proper notation.

8. You must use calculus methods to solve the problem, and you must justify that your answer really does give the maximum area.

   Find the length and width of the rectangle with the largest area whose lower two vertices lie on the x-axis and whose upper two vertices lie on the graph of $y = 18 - x^2$.

Graph of $y = 18 - x^2$



Objective Function:

$A(x, y) = 2x \cdot y$

Constraint: $y = 18 - x^2$

re-write $A(x,y)$ as $A(x)$

$A(x) = 2x(18 - x^2)$

The critical #s of $A(x)$ are: $A'(x) = 0$ or DNE

$A'(x) = 36 - 6x^2 = 0 \Rightarrow x = \sqrt{6}$ ($x = -\sqrt{6}$ is NOT in domain)

Justify $x = \sqrt{6}$ gives the MAX area:

$A''(x) = -12x$ (always negative) ($A(x)$ concave down)

length $\Rightarrow 2x = 2\sqrt{6}$

width $\Rightarrow y = 18 - x^2 = 18 - (\sqrt{6})^2 = 12$

9. The edges of a cube increase at a rate of 2 cm/sec. How fast is the volume changing when the length of each edge is 50 cm? You must include correct units as part of your answer.

Let x be the edge of a cube

Given: $\frac{dx}{dt} = \frac{2\,cm}{sec.}$, $x = 50\,cm$. at the instant

Asked: $\frac{dV}{dt} = ?$ when $x = 50\,cm$.

Volume of a cube $(V(x) = x^3)$

$V(x) = x^3$

$$\frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt} = 3 \cdot 50^2 \cdot 2 = 15{,}000 \frac{cm^3}{sec.}$$

The volume is increasing at a rate of 15,000 cm^3/sec.

10. (a) **(5 points)** The selling price per unit of a certain product is modeled by $p(x) = \dfrac{800}{x+5}$ if x units are already being produced. Use marginal analysis to estimate the revenue derived from producing the 16th unit. Your answer should be an integer or a simplified fraction of integers.

$$R(x) = p(x)\cdot x = \frac{800x}{x+5}$$

$$R'(x) = \left(\frac{800x}{x+5}\right)' = \frac{800(x+5) - 800x\cdot 1}{(x+5)^2} = \frac{800\cdot 5}{(x+5)^2}$$

$$R'(15) = \frac{800\cdot 5}{(15+5)^2} = \frac{\overset{2}{\cancel{800}}\cdot 5}{\cancel{20\cdot 20}} = \$10$$

(b) **(5 points)** Evaluate the limit or determine that it does not exist. If the limit is infinite, then your answer should be $+\infty$ or $-\infty$ (as appropriate), instead of "does not exist".

$$\lim_{x\to\infty}\left(\frac{3x^4+2x+2}{x^2-25}\right)$$

$$\lim_{x\to\infty}\frac{\frac{3x^4+2x+2}{x^2}}{\frac{x^2-25}{x^2}} = \lim_{x\to\infty}\frac{3x^2+\cancel{\frac{2}{x}}^{\to 0}+\cancel{\frac{2}{x^2}}^{\to 0}}{1-\cancel{\frac{25}{x^2}}^{\to 0}}$$

$$\lim_{x\to\infty}\frac{3x^2}{1}=\infty$$

$\boxed{\text{OR}}$

$$\lim_{x\to\infty}\frac{3x^4+2x+2}{x^2-25} \overset{\text{"DSP"}}{=} \frac{\infty}{\infty} \Rightarrow \overset{LR}{=} \lim_{x\to\infty}\frac{12x^3+2}{2x}$$

$$\overset{\text{"DSP"}}{=} \frac{\infty}{\infty} \Rightarrow \overset{LR}{=} \lim_{x\to\infty}\frac{36x^2}{2} \overset{\text{"DSP"}}{=} \infty$$

11. Find an equation of the line normal to the graph of the equation $7\sqrt{x} - 3\sqrt{y} = y - 4$ at $(4, 9)$.

Implicitly differentiate the equation with respect to x to obtain:

$$\frac{7}{2} \cdot x^{-1/2} - \frac{3}{2} \cdot y^{-1/2} \cdot \frac{dy}{dx} = \frac{dy}{dx} \qquad \frac{7}{2\sqrt{x}} = \left(\frac{3}{2\sqrt{y}} + 1\right) \frac{dy}{dx}$$

Substituting the point (4, 9) gives:

$$\frac{7}{2} \cdot 4^{-1/2} - \frac{3}{2} \cdot 9^{-1/2} \cdot \frac{dy}{dx} = \frac{dy}{dx}$$

$$\frac{7}{2} \cdot \frac{1}{2} - \frac{3}{2} \cdot \frac{1}{3} \cdot \frac{dy}{dx} = \frac{dy}{dx}$$

$$\frac{7}{4} - \frac{1}{2} \cdot \frac{dy}{dx} = \frac{dy}{dx}$$

$$\frac{7}{4} = \frac{3}{2} \cdot \frac{dy}{dx} \Rightarrow \underbrace{\frac{7}{6} = \frac{dy}{dx} = \frac{14}{12}}_{\text{both acceptable}}$$

$m_{tan} = \frac{7}{6}$, $m_{normal} = \frac{-6}{7}$

Equation of the normal line to the graph at (4,9):

$$\boxed{y - 9 = \frac{-6}{7}(x - 4)}$$

For this problem, you are not required to show work, and any work you do write will not be graded. Write your final answers in the table below. You may use the back of this page for scratch work. Each row in the table is worth 2.5 points, graded with no partial credit.

12. Consider the function f and its derivatives below.

$$f(x) = \frac{x^3}{x^2-4} \quad , \quad f'(x) = \frac{x^2(x^2-12)}{(x^2-4)^2} \quad , \quad f''(x) = \frac{8x(x^2+12)}{(x^2-4)^3}$$

Find where f is increasing, decreasing, concave up, and concave down. Use interval notation in your answers. Also, find all horizontal and vertical asymptotes, relative extrema, and inflection points. Write "NONE" as your answer, if appropriate.

Domain of f	$(-\infty,-2),(-2,2),(2,\infty)$ AND, U
Increasing	$(-\infty,-2\sqrt{3}),\ (2\sqrt{3},\infty)$
Decreasing	$(-2\sqrt{3},-2),(-2,0),(0,2),(2,2\sqrt{3})$ OK $(-2\sqrt{3},2\sqrt{3})$ OK.
Concave up	$(-2,0),(2,\infty)$
Concave down	$(-\infty,-2),(0,2)$
Equations of horizontal asymptotes	NONE
Equations of vertical asymptotes	$X=2,\ X=-2$
x-coordinates of relative maxima	$X=-2\sqrt{3}$
x-coordinates of relative minima	$X=2\sqrt{3}$
x-coordinates of inflection points	$X=0$