Note Title 1/30/2020

2.2. Algebraic Computation of Limits

Direct Substitution Property (DSP)

If $\lim_{x \to c} f(x) = L = f(c)$

Then $f(x)$ has the DSP (at $x=c$)

when $f(c)$ is defined, the limit can be found simply by using DSP.

If not, we use algebra to evaluate limits.

Exp1) Evaluate $\lim_{x \to 2} f(x) = \lim_{x \to 2} \underbrace{(2x^5 - 9x^3 + 3x^2 - 11)}_{\text{polynomials always use DSP.}}$

Solution: $\lim_{x \to 2} f(x) = f(2) = 2 \cdot 2^5 - 9 \cdot 2^3 + 3 \cdot 2^2 - 11$

$= 64 - 9 \cdot 8 + 3 \cdot 4 - 11$

$= 64 - 72 + 12 - 11$

$= -8 + 1 = \boxed{-7}$

Exp2) Evaluate $\lim_{z \to -1} f(z)$. $\left(f(z) = \dfrac{z^3 - 3z + 7}{5z^2 + 9z + 6} \right)$

Solution: Is $z=-1$ in the domain of $f(z)$ Yes!

$$5(-1)^2+9(-1)+6 = \underbrace{5-9+6}_{} = 11-9 = 2 \neq 0$$

$$\lim_{z\to -1} \frac{z^3-3z+7}{5z^2+9z+6} = f(-1) = \frac{(-1)^3-3(-1)+7}{2}$$

$$= \frac{-1+3+7}{2} = \boxed{\frac{9}{2}}$$

Exp3) Evaluate $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

Solution: Is $x=0$ in the domain of $\frac{\sqrt{x+1}-1}{x}$? NO!

$x=0$ "division by 0"

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \cdot \left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$$

Recall:

$(a-b)(a+b)=a^2-b^2$

$$= \lim_{x\to 0} \frac{(\sqrt{x+1})^2-1^2}{x\cdot(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{\cancel{x}\cancel{+1}-\cancel{1}}{\cancel{x}(\sqrt{x+1}+1)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} \overset{\text{"DSP"}}{=} \frac{1}{\sqrt{0+1}+1} = \boxed{\frac{1}{2}}$$

You try it!

Evaluate $\lim\limits_{x\to 2} \dfrac{\overbrace{x^2-4x+4}^{f(x)}}{x^2-x-2}$.

Is $x=2$ in the domain of $f(x)$? No!

$2^2-2-2=0$ "div. by 0"

Use factoring to evaluate limit.

$$\lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x+1)}=\lim_{x\to 2}\frac{\cancel{(x-2)}(x-2)}{\cancel{(x-2)}(x+1)}\overset{\text{"DSP"}}{=}\frac{2-2}{2+1}=\frac{0}{3}=0$$

Exp4) Evaluate $\lim\limits_{x\to 0}(\sin x)^2$.

Solution: Is $x=0$ in the domain of $(\sin x)^2$? YES

$$\lim_{x\to 0}(\sin x)^2=\underbrace{\left(\lim_{x\to 0}(\sin x)\right)^2}_{\text{Power Rule}}\overset{\text{"DSP"}}{=}(\sin 0)^2$$

$$=0^2=0$$

Special Trig. Limits

① $\lim\limits_{h\to 0}\left(\frac{\sin h}{h}\right)=1$ ② $\lim\limits_{h\to 0}\left(\frac{h}{\sin h}\right)=1$

E.g: $\lim\limits_{x\to 0}\left(\frac{\sin(4x)}{4x}\right)=1$

$(x\to 0\ ;\ 4\cdot x\to 0)$

E.g: $\lim\limits_{x\to 0}\frac{(-2x)}{\sin(-2x)}=1$

E.g: $\lim\limits_{x\to 0}\left(\frac{\sin\left(\frac{e^7}{2\pi}\cdot x\right)}{\frac{e^7}{2\pi}\cdot x}\right)=1$

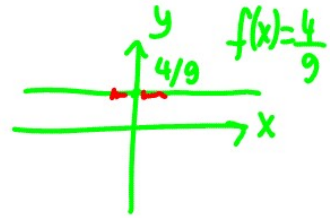
Exp5) Evaluate $\lim\limits_{x\to 0}\frac{\sin(4x)}{9x}$.

Solution: $\lim\limits_{x\to 0}\left(\frac{\sin(4x)}{9x}\cdot\frac{4}{4}\right)=\lim\limits_{x\to 0}\left(\frac{\sin(4x)\cdot 4}{\underbrace{9x\cdot 4}}\right)$

$9x\cdot 4=4x\cdot 9$

$=\lim\limits_{x\to 0}\left(\boxed{\frac{\sin(4x)}{4x}}\cdot\frac{4}{9}\right)$

$$= \underbrace{\lim_{x\to 0}\left(\frac{\sin(4x)}{4x}\right)}_{1} \cdot \underbrace{\lim_{x\to 0}\left(\underbrace{\frac{4}{9}}_{f(x)}\right)}_{\frac{4}{9}\ \text{(constant rule)}}$$



$$= 1 \cdot \frac{4}{9} = \frac{4}{9}$$

Be careful!

$$\lim_{x\to 0}\left(\frac{\sin(4x)}{9x}\right)\cdot \cancel{\frac{9}{9}} \quad \text{vs.} \quad \lim_{x\to 0}\left(\frac{\sin(6x)}{9x}\right)\cdot\frac{2}{2} \quad \text{OR} \quad \cdot\frac{6}{6}$$

$$\lim_{x\to 0} \frac{\sin(6x)\cdot 2}{6x\cdot 3} = \frac{2}{3}$$

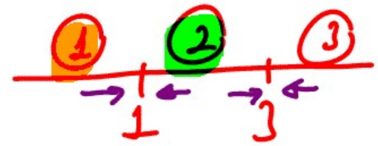
Limits of Piecewise-Defined Functions

To evaluate $\lim_{x\to c} f(x)$ where the domain of f is divided into pieces, we first look to see whether c is a value separating the pieces. If so, we need to consider one-sided limits.

Exp6) Evaluate $\lim_{x\to 3} f(x)$, $\lim_{x\to 1} f(x)$ where:

$$f(x) = \begin{cases} 2(x+1), & \text{if } x<1 \quad ① \end{cases}$$

$$\begin{cases} 4, & \text{if } 1<x\leq 3 \quad (2) \\ x^2-1, & \text{if } x>3 \quad (3) \end{cases}$$



$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2(x+1) = 2(1+1) = 4$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4) = 4$$

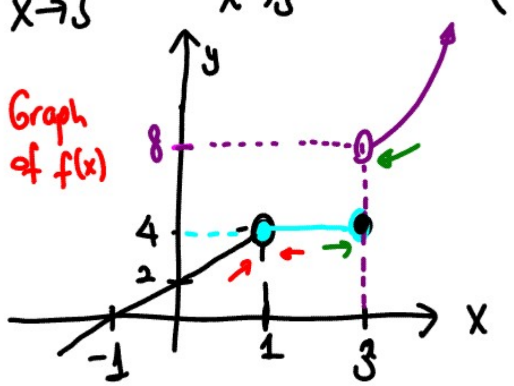
$$\left.\lim_{x\to 1^-} f(x) = 4 = \lim_{x\to 1^+} f(x)\right\} \lim_{x\to 1} f(x) = 4$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 4 = 4$$

Is $4=8$? NO!

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x^2-1) \overset{\text{"DSP"}}{=} 3^2-1 = 9-1 = 8$$

$$\left.\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x) \quad (4\neq 8)\right\} \lim_{x\to 3} f(x) \text{ DNE}$$

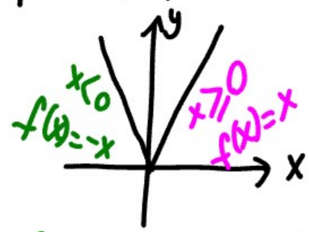


$$f(x) = \begin{cases} 2(x+1), & x<1 \\ 4, & 1<x\leq 3 \\ x^2-1, & x>3 \end{cases}$$

Exp7) Evaluate $\lim_{x\to 4} \underbrace{\left(\frac{|x-4|}{x-4}\right)}_{g(x)}$

Review

$f(x) = |x|$



$$f(x) = |x-4| \begin{cases} \text{if } x \geq 4, f(x) = (x-4) \ ① \\ \text{if } x < 4, f(x) = -(x-4) \ ② \end{cases}$$

$$f(x) = \begin{cases} \text{if } x \geq 0, f(x) = x \\ \text{if } x < 0, f(x) = -x \end{cases}$$

② $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} \frac{-\cancel{(x-4)}}{\cancel{x-4}} = -1$

① $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} \frac{(x-4)}{x-4} = 1$

$\lim_{x\to 4} g(x)$ DNE $(-1 \neq 1)$

Exp8) Evaluate $\lim_{t\to 0} \frac{\tan 5t}{\tan 2t}$.

Recall:

#1) $\lim_{t\to 0} \frac{\sin t}{t} = 1$

#2) $\tan t = \frac{\sin t}{\cos t}$

#3) $\sin 0 = 0$

$\cos 0 = 1$

Solution: Is $t=0$ in the domain of $\frac{\tan 5t}{\tan 2t}$?

No! $\tan(2\cdot 0) = \tan 0 = 0$ "div. by 0"

Use trigonometric identities to re-write:

$$\lim_{t\to 0} \frac{\frac{\sin 5t}{\cos 5t}}{\frac{\sin 2t}{\cos 2t}} = \lim_{t\to 0}\left[\left(\frac{\sin 5t}{\cos 5t} \cdot \frac{\cos 2t}{\sin 2t}\right) \cdot \frac{5t}{5t} \cdot \frac{2t}{2t}\right]$$

$$= \lim_{t\to 0}\left[\frac{\sin 5t}{5t} \cdot 5t \cdot \frac{1}{\cos 5t} \cdot \frac{\cos 2t}{\sin 2t} \cdot \frac{2t}{2t}\right]$$

$$= \lim_{t\to 0}\left[\cancel{\frac{\sin 5t}{5t}}^{\to 1}\cdot 5t\cdot\frac{1}{\cos 5t}\cdot\cos 2t\cdot\cancel{\frac{2t}{\sin 2t}}^{\to 1}\cdot\frac{1}{2t}\right]$$

$$= \lim_{t\to 0}\left[\frac{5\cancel{t}\cdot\cos 2t}{\cos 5t\cdot 2\cancel{t}}\right] \overset{\text{"DSP"}}{=} \frac{5\cdot\cancel{\cos(2\cdot 0)}^{\to 1}}{\cancel{\cos(5\cdot 0)}^{1}\cdot 2} = \frac{5}{2}$$