Note Title 2/3/2020

2.3. Continuity

We know the vague definition of continuity as "being able to graph a function w/out breaks".

Calculus Def: f is continuous at $x=c$ if:

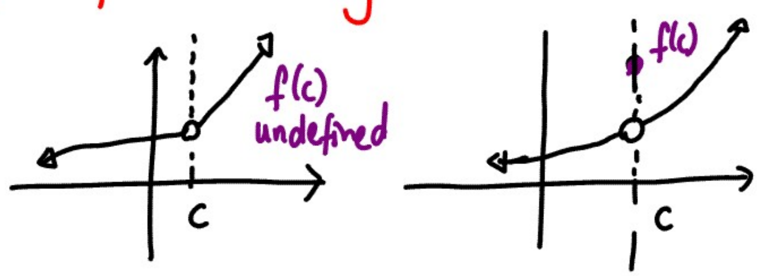
1) $f(c)$ is defined $(x=c)$

2) $\lim_{x\to c} f(x)$ exists $(x\neq c)$

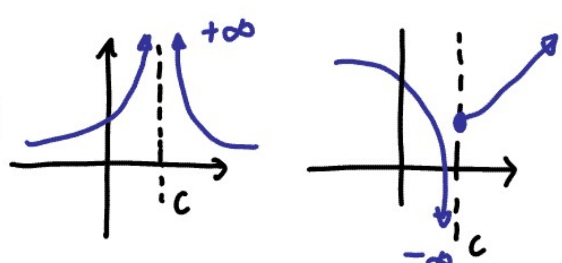
3) $\underbrace{\lim_{x\to c} f(x) = f(c)}_{\text{"DSP"}}$ (Whole pic.)

Types of Discontinuity

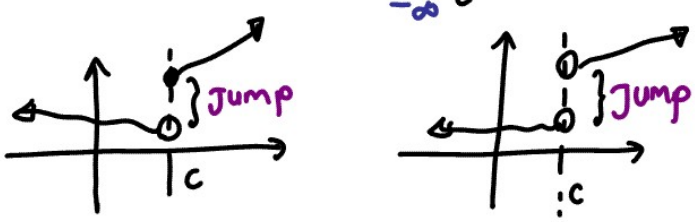
① Hole



② Pole (Infinite)



③ Jump



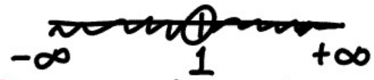
④ Essential

Expl) Where is $\underbrace{f \text{ continuous?}}$ $f(x) = \dfrac{x^2+2x-3}{x-1}$ ← check the denominator

Solution:

$x-1=0$

$x=1$ (Div. by 0) (Hole at $x=1$)



[,] → incl.

(,) → excl.

interval notation (may use U or ,)

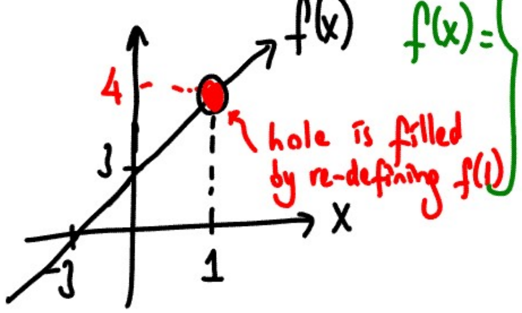
$\underbrace{\begin{cases} (-\infty, 1) \cup (1, \infty) \\ (-\infty, 1), (1, \infty) \end{cases}}_{f \text{ is continuous on}}$

Can we redefine the value of f at $x=c$ $(c=1)$

So that f becomes continuous at $x=c$?

$f(x) = \dfrac{x^2+2x-3}{x-1} = \dfrac{(x+3)\cancel{(x-1)}}{\cancel{x-1}}$; $x \neq 1$

$f(x) = \begin{cases} f(x) = \dfrac{x^2+2x-3}{x-1}, & \text{if } x \neq 1 \\ f(x) = x+3, & \text{if } x = 1 \end{cases}$



Exp2) Where's g continuous?

$$g(x)=\begin{cases} x^2-2, & \text{if } x>0 \\ 2x^4-2\cdot\cos x, & \text{if } x\leq 0 \end{cases}$$

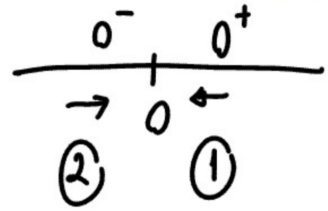
Recall ALL polynomials are continuous in their domain.

Check each "piece" of the "whole" function at transition (suspicious) points. (e.g: $x=0$)

check for a possible point of discontinuity at $x=0$.

$$g(x)=\begin{cases} x^2-2, & \text{if } x>0 \\ 2x^4-2\cdot\cos x, & \text{if } x\leq 0 \end{cases}$$

① $g(0)=2\cdot 0^4-2\cdot\cos 0$
$= 0-2\cdot 1$
$= -2$



② $\lim_{x\to 0} g(x) = \cancel{?}$ -2

$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} (2x^4-2\cdot\cos x) \overset{\text{"osp"}}{=} -2$
(Cos 0 = 1)

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (x^2-2) \overset{\text{"osp"}}{=} 0^2-2=-2$

$-2=-2$

$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x) = -2 \Rightarrow \lim_{x\to 0} g(x) = -2$

(3) $g(0) = \lim\limits_{x \to 0} g(x) = -2$

Yes, the function $g(x)$ is continuous at $x=0$.

Continuous on $(-\infty, +\infty)$

Exp3) Where is f continuous? Write in interval notation.

$$f(x) = \begin{cases} 3 - x^2, & \text{if } -5 \leq x < 2 \\ x^3 - 8, & \text{if } 2 \leq x < 5 \end{cases}$$

Solution: All polynomials are cont. in their domain.
check the transition point $(x=2)$

(1) $f(2) = 2^3 - 8$
$= 8 - 8 = 0$

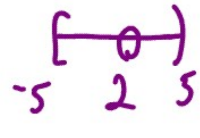
(2) $\lim\limits_{x \to 2} f(x) = \cancel{?}$ DNE

$$\lim_{x \to 2^-} (3 - x^2) \overset{\text{"DSP"}}{=} 3 - 2^2 = 3 - 4 = -1$$

$$\lim_{x \to 2^+} (x^3 - 8) \overset{\text{"DSP"}}{=} 2^3 - 8 = 0$$

Since:
$\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x)$
$-1 \neq 0$
$\lim\limits_{x \to 2} f(x)$ DNE

Point of discontinuity at $x=2$.



$f(x)$ is Continuous on $\underbrace{[-5, 2), (2, 5)}_{\text{excluding } x=2}$

Exp4) Find the values of A and B which make $f(x)$ continuous for all x; or explain why such values do not exist.

$$f(x) = \begin{cases} Ax+3, & \text{if } x<1 & \text{linear} \\ 5, & \text{if } x=1 & \text{constant} \\ x^2+B, & \text{if } x>1 & \text{quadratic} \end{cases}$$

continuous

Solution: All polynomials are continuous.
Identify transition P. $(x=1)$

① $f(1)$ ② $\lim\limits_{x\to1} f(x)$ ③ ① $\overset{?}{=}$ ②

① $f(1)=5$

② $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-} (Ax+3) \overset{\text{"DSP"}}{=} A\cdot1+3 = A+3$

$\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} (x^2+B) \overset{\text{"DSP"}}{=} 1^2+B = 1+B$

$\lim\limits_{x\to1} f(x)$ exists : $\left.\begin{array}{l} \lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^+} f(x) \\ A+3 = 1+B \end{array}\right\}$

③ ① = ② $\}$ $f(1) = \lim\limits_{x\to1} f(x)$

$$\underbrace{5 = A+3}_{} = 1+B$$

$5 = A+3 \Rightarrow \boxed{A=2}$

$5 = 1+B \Rightarrow \boxed{B=4}$

Exp5)

$$f(x) = \begin{cases} \dfrac{\sin(Ax)}{x}, & \text{if } x<0 \quad 0^- \\ 5, & \text{if } x=0 \\ x^3+B, & \text{if } x>0 \quad 0^+ \end{cases}$$

Find the values of A and B which make f(x) continuous.

Recall: $\lim\limits_{x\to 0} \dfrac{\sin(Ax)}{(Ax)} = 1$

Transition point: $x=0$

① $f(0) = 5$

② $\lim\limits_{x\to 0^-} f(x) = ?$ $\quad \lim\limits_{x\to 0^+} f(x) = ?$

$\lim\limits_{x\to 0^-} \dfrac{\sin(Ax)}{x} \overset{\text{"DSP"}}{=} \text{"}\dfrac{0}{0}\text{"}$ indeterminate form

$$\lim_{x\to 0^-} \frac{\sin(Ax)}{x} \cdot \frac{A}{A} = \lim_{x\to 0^-} \underbrace{\frac{\sin(Ax)}{Ax}}_{1} \cdot A$$

$$= \lim_{x\to 0^-} A = A$$

$$\lim_{x \to 0^+} (x^3 + B) = 0^3 + B = B$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$A = B$$

(3) In order for $f(x)$ to be continuous at $x=0$;

$$\lim_{x \to 0} f(x) = f(0).$$

$$A = B = 5$$

The values of $A = B = 5$ make f continuous in its domain.

Exp6) Find the value of constant B which make $G(x)$ continuous for all x; or explain why such values do not exist.

$$G(x) = \begin{cases} B, & \text{if } x \leq 1 \\ \dfrac{2\sqrt{x} - 2}{x - 1}, & \text{if } x > 1 \end{cases}$$

Solution: The 2nd "piece" of $G(x)$ $\left(\dfrac{2\sqrt{x} - 2}{x - 1} \right)$ is not continuous at $x=1$, however, $x=1$ is NOT in the domain of this 2nd "piece" since it is defined when $x>1$.

Therefore, both pieces are considered to be continuous in their domains.

check for continuity at $x=1$ (transition point)

① $G(1)=B$

② $\lim\limits_{x\to 1^-} G(x) = \lim\limits_{x\to 1^-} B = B$

$$\lim_{x\to 1^+} G(x) = \lim_{x\to 1^+}\left(\frac{2\sqrt{x}-2}{x-1}\right)$$

Recall:

$(\sqrt{x}-1)\cdot(\sqrt{x}+1) = x-1$

$$= \lim_{x\to 1^+}\left(\frac{2(\sqrt{x}-1)}{x-1}\cdot\frac{(\sqrt{x}+1)}{(\sqrt{x}+1)}\right)$$

$$= \lim_{x\to 1^+} \frac{2\cancel{(x-1)}}{\cancel{(x-1)}(\sqrt{x}+1)} = \lim_{x\to 1^+}\left(\frac{2}{\sqrt{x}+1}\right)$$

$$\overset{\text{"DSP"}}{=} \frac{2}{\sqrt{1}+1} = \frac{2}{2} = 1$$

③ $\lim_{x \to 1} G(x)$ must be equal to $G(1)$

for $G(x)$ to be continuous at $x=1$.

$$\lim_{x \to 1^-} G(x) = \lim_{x \to 1^+} G(x) = G(1)$$

$$\boxed{B = 1}$$