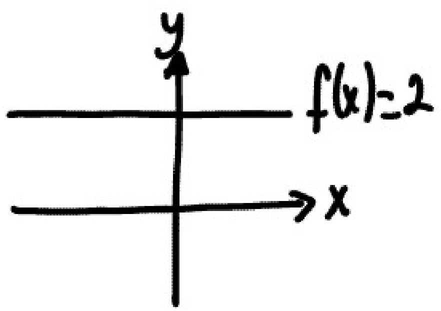
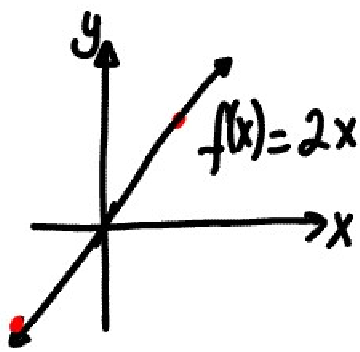
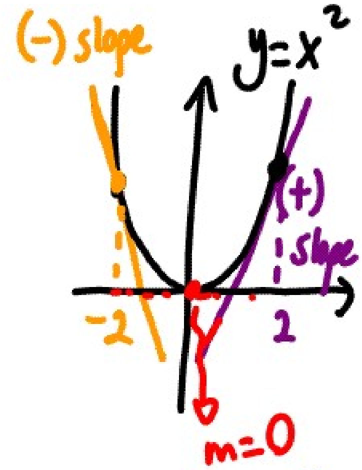
Chapter 3 – Differentiation

Note Title 2/6/2020

Motivation: How do we compute the rate of change for functions by using Calculus?







slope = 0 at every x-value

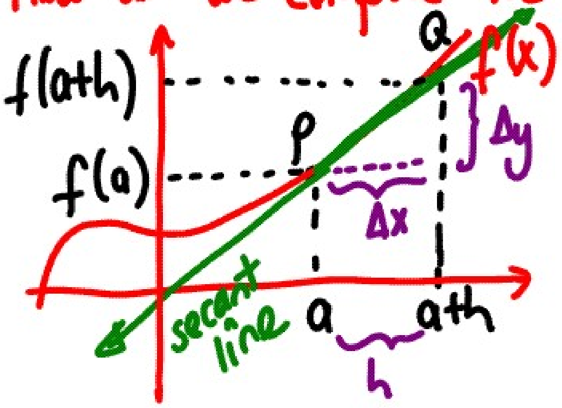
slope = 2 at every x-value

(slope changes at each x-value)

$\underbrace{\qquad\qquad}$ Algebra was sufficient

$\underbrace{\qquad\qquad}$ We need Calculus!

How do we compute the slope of the tangent line at P?



$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$

$$P(a, f(a)) \qquad Q(a+h, f(a+h))$$

(slope of secant line) $m_{sec} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(a+h) - f(a)}{\cancel{(a}+h) - \cancel{a}} = \dfrac{f(a+h) - f(a)}{h}$

As we move Q towards P on the graph ($h \to 0$); we obtain m_{tan} by:

(slope of tangent line) $m_{tan} = \lim\limits_{h \to 0} m_{sec} \Rightarrow m_{tan} = \lim\limits_{h \to 0} \left(\dfrac{f(a+h) - f(a)}{h} \right)$

Def: The slope of the tangent line to the graph of f at point $P(a, f(a))$ is:

$$m_{tan} = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

The equation of the tangent line is:

$$y - f(a) = m_{tan} (x - a)$$

(point-slope formula)

Def: Slope of a tangent line to a graph at a point

The tangent line to the graph of $f(x)$ at point $P(a, f(a))$:

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

(slope of a tangent line)

Equation of the tangent line to $f(x)$ at P:

$y - y_1 = m(x - x_1)$

$P(a, f(a))$

(x_1, y_1)

$y - f(a) = m_{tan}(x-a)$

Def: The derivative of f at $x=a$ is:

$$f'(a) = \lim_{h \to 0} \underbrace{\frac{f(a+h)-f(a)}{h}}_{m_{sec}} = m_{tan}$$

read as "f prime of a"

Exp1) Find the equation of the tangent line to $f(x) = x^2 - 1$ at $x = 2$.

Solution:

$f(x) = x^2 - 1$

$f(a) = f(2) = 2^2 - 1$

$f(2) = 3$ $\quad P(2,3)$

$f(2+h) = (2+h)^2 - 1$

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$m_{tan} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$m_{tan} = \lim_{h \to 0} \frac{(4+4h+h^2-1) - 3}{h}$$

$$= \lim_{h \to 0} \left(\frac{\cancel{3} + 4h + h^2 - \cancel{3}}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\cancel{h}(4+h)}{\cancel{h}} \right)$$

$$= \lim_{h \to 0} (4+h) \overset{\text{"dsr"}}{=} 4$$

Eq. of tangent line is:

$y - y_1 = m_{tan}(x - x_1)$

$$\boxed{y - 3 = 4(x - 2)}$$

You try it!

Find the equation of the tangent line to $f(t) = 4 - t^2$ at $t = 0$.

Solution:

$f(t) = 4 - t^2$

$f(a) = f(0) = 4 - 0^2 = 4$

$P(0,4)$

$f(a+h) = f(0+h) = f(h)$

$f(h) = 4 - h^2$

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{4} - h^2) - \cancel{4}}{h}$$

$$= \lim_{h \to 0} \left(\frac{-h^2}{h}\right) = \lim_{h \to 0} (-h)$$

$$\overset{\text{"DSP"}}{=} 0$$

eq. of the tangent line to $f(t)$ at $t = 0$:

$m_{tan} = 0$

$P(0,4)$
(x_1, y_1)

$$y - y_1 = m_{tan}(x - x_1)$$

$$y - 4 = 0 \cdot (x - 0)$$

$$\boxed{y - 4 = 0}$$

Exp2) Calculate $f'(0)$ when $f(x)=|x|$

"Find the slope of the tangent line to $f(x)$ at $x=0$"

Solution:

$$\boxed{f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}=\lim_{h\to 0}\frac{|h|-0}{h}$$

$f(x)=|x|$

$a=0$

$f(a+h)=f(h)=|h|$

$f(a)=f(0)=|0|=0$

$$=\underbrace{\lim_{h\to 0}\frac{|h|}{h}}$$

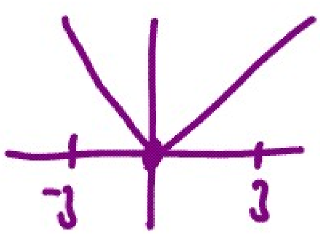
$h\to 0$ LL 0 RL

Recall:

$f(x)=|x|\begin{cases} f(x)=x, & x\geq 0 \\ f(x)=-x, & x<0 \end{cases}$

$$\left.\begin{aligned} &\lim_{h\to 0^-}\frac{|h|}{h}=\lim_{h\to 0^-}\frac{-h}{h}=\lim_{h\to 0}(-1)=-1 \\ &\lim_{h\to 0^+}\frac{|h|}{h}=\lim_{h\to 0^+}\frac{h}{h}=\lim_{h\to 0^+}(1)=1 \end{aligned}\right\}\ -1\neq 1$$

$\underbrace{f'(0)\text{ DNE}}$



$f'(3)=1$

$f'(-3)=-1$

$f(x)=|x|$ is differentiable at every point EXCEPT $x=0$.

Exp3) Calculate $f'(x)$ when $f(x)=\sqrt{x}$

Solution: $f(x)=\sqrt{x}$ is defined for $x \geq 0$

$[0,\infty)$

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$ At $x=a$

$$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$ General Formula

$f(x)=\sqrt{x}$

$f(x+h)=\sqrt{x+h}$

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$

Recall:

$(a-b)(a+b)=a^2-b^2$

$$=\lim_{h\to 0}\frac{(\cancel{x}+\cancel{h})-\cancel{x}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \overset{\text{"DSP"}}{=} \frac{1}{\sqrt{x+0}+\sqrt{x}}$$

$$f'(x)=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

$f(x)$ is $[0,\infty)$

$f'(x)$ is $(0,\infty)$

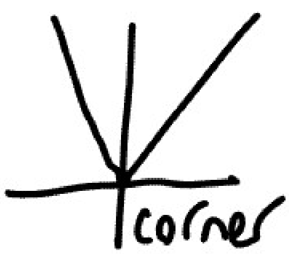
* $f(x) = \sqrt{x}$ is defined on $[0, \infty)$

however, it's NOT differentiable at $x=0$. $f(x)$ is differentiable on $(0, \infty)$

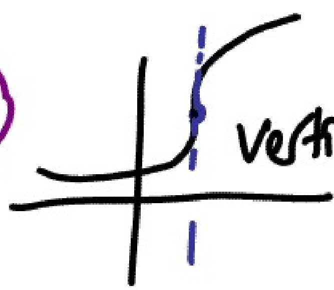
If $f'(a)$ exist,

Then f is differentiable at $x=a$.

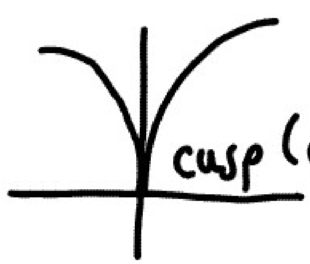
When does a function NOT differentiable?

① 

$f(x) = |x|$ continuous f.

② 

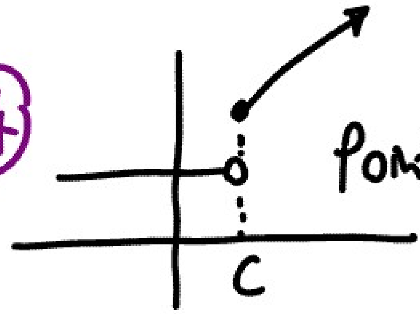
$f(x) = x^{1/3}$ continuous f.

③ 

$f(x) = x^{2/3}$ continuous f.

② & ③ Power functions where $f(x)=x^n$ $(0<n<1)$

④



Point of discontinuity $(x=c)$

A function differentiable is continuous, however, a continuous function does not have to be differentiable at every point in its domain (e.g: $f(x)=|x|$)