Note Title 2/10/2020

# 3.2-3.3. Techniques of Differentiation

Recall that: $\underbrace{f'(x)}_{m_{tan}} = \lim_{h\to 0} \underbrace{\frac{f(x+h)-f(x)}{h}}_{m_{sec}}$

$$f'(x)\Big|_{x=a} = f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

OR

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

## Table of Derivative Rules (Memorize)

| Type of Function | $f(x)$ | $f'(x)$ |
|---|---|---|
| constant | $c$ | $0$ e.g.: $\pi$, $e^{100}$, $\ln 4$ |
| power | $x^n$ | $n\cdot x^{n-1}$ |
| exponential | $e^x$ | $e^x$ |
| natural log | $\ln x$ | $\frac{1}{x}$ |

Trigonometric Functions

| | |
|---|---|
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\tan x$ | $\sec^2 x$ |
| $\cot x$ | $-\csc^2 x$ |
| $\csc x$ | $-\csc x \cdot \cot x$ |
| $\sec x$ | $\sec x \cdot \tan x$ |

Table for Procedural Derivative Rules

Memorize

| Type of op. | $f(x)$ | $f'(x)$ |
|---|---|---|
| Addition | $f+g$ | $f'+g'$ |
| Subtraction | $f-g$ | $f'-g'$ |
| multiplication (Product Rule) | $f \cdot g$ | $f' \cdot g + f \cdot g'$ |
| Division (Quotient Rule) | $\frac{f}{g}$ | $\frac{f' \cdot g - f \cdot g'}{g^2}$ |

# Higher-Order Derivatives

| # Derivative | Leibniz Notation | Lagrange Notation |
|---|---|---|
| 1st Der. | $df/dx$ | $f'$ ("f prime") |
| 2nd Der. | $d^2f/dx^2$ | $f''$ |
| 3rd Der. | $d^3f/dx^3$ | $f'''$ |
| 4th Der. | $d^4f/dx^4$ | $f^{(4)}$ |
| ⋮ | ⋮ | ⋮ |
| Nth Der. | $d^Nf/dx^N$ | $f^{(N)}$ |

$$f^{(4)} \neq f^4$$

4th Derivative of $f(x)$ ≠ 4th Power of $f(x)$

Expl) Verify $\frac{d}{dx}(\tan x) = \sec^2 x = (\sec x)^2 = \sec(x)^2$

$\neq \sec x^2$

$\neq \sec(x^2)$

Use trig. identity

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

Recall:

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

Use quotient rule

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{(\cos x)^2}$$

$$= \frac{\cos x \cdot \cos x + \sin x \cdot (+\sin x)}{(\cos x)^2}$$

Recall from Trig:

$\cos^2 x + \sin^2 x = 1$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Recall:

$\frac{1}{\cos x} = \sec x$

$$= \left(\frac{1}{\cos x}\right)^2 = \sec^2 x$$

Exp2) Calculate $f'(x)$ when $f(x) = \left(\frac{1}{2\sqrt{x}} + \frac{x^2}{4} - \pi^{2/3}\right)$

$$f'(x) = \left(\frac{1}{2\sqrt{x}}\right)' + \left(\frac{x^2}{4}\right)' + \underbrace{\left(-\pi^{2/3}\right)'}_{\text{constant}}$$

$$= \left(\frac{1}{2}\cdot x^{-1/2}\right)' + \left(\frac{1}{4}\cdot x^2\right) + 0$$

Recall:

$(x^n)' = n\cdot x^{n-1}$

$$= \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot x^{-1/2-1} + \frac{1}{4}\cdot 2x$$

$$= -\frac{1}{4}\cdot x^{-3/2} + \frac{x}{2}$$

Exp3) Calculate $g'(x)$ when $g(x) = \frac{4x-7}{3-x^2} + \underbrace{86}_{\text{constant}}$

$$g'(x) = \frac{(4x-7)'\cdot(3-x^2) - (4x-7)\cdot(3-x^2)'}{(3-x^2)^2} + 0$$

$$= \frac{4(3-x^2) - (4x-7)\cdot(-2x)}{(3-x^2)^2}$$

$$= \frac{12-4x^2+8x^2-14x}{(3-x^2)^2} = \frac{4x^2-14x+12}{(3-x^2)^2}$$

Exp4) Calculate $\frac{d}{dx}\left(\frac{2x^5-3x^2+11}{x^3}\right)$

Simplify then differentiate

$$\frac{d}{dx}\left(\frac{2x^5}{x^3} - \frac{3x^2}{x^3} + \frac{11}{x^3}\right)$$

$$\frac{d}{dx}\left(2x^2 - \frac{3}{x^1} + \frac{11}{x^3}\right) = \frac{d}{dx}\left(2x^2 - 3\cdot x^{-1} + 11\cdot x^{-3}\right)$$

$$= 4x + 3x^{-2} - 33x^{-4}$$

$$= 4x + \frac{3}{x^2} - \frac{33}{x^4}$$

Exp5) Calculate $f'(x)$ when $f(x) = e^x \cdot \cos x + \dfrac{\sqrt{x} \cdot \ln x}{x^3}$

$$f(x) = e^x \cdot \cos x + \frac{x^{1/2} \cdot \ln x}{x^3} = \underbrace{e^x \cdot \cos x}_{\text{product rule}} + x^{1/2 - 3} \cdot \ln x \quad \left( \frac{\ln x}{x^3} \neq \ln\left(\frac{x}{x^3}\right) \right)$$

$$f'(x) = \underbrace{e^x \cdot \cos x + e^x \cdot (-\sin x)}_{} + \underbrace{\left(x^{-5/2} \cdot \ln x\right)'}_{\text{product rule}}$$

$$= e^x \cdot \cos x - e^x \cdot \sin x + \left( \frac{-5}{2} \cdot x^{-7/2} \cdot \ln x + x^{-5/2} \cdot \frac{1}{x} \right) \quad \left( (\ln x)' = \frac{1}{x} \right)$$

$$= e^x (\cos x - \sin x) - \frac{5}{2} \cdot x^{-7/2} \cdot \ln x + x^{-7/2}$$

Recall:

$$\frac{x^{-5/2}}{x^1} = x^{-5/2 - 1} = x^{-\frac{5}{2} - \frac{2}{2}}$$
$$= x^{-7/2}$$

Exp6) Find the equation of the tangent line to $f(x)=\dfrac{x^2+5}{x+5}$ at $x=1$. Any form of equation is acceptable.

Solution: First find $m_{tan}\Big|_{x=1} = f'(1) = ?$

Then use the point $(x_1,y_1)=(1,f(1))$ to write the equation of the tangent line to $f(x)$ at $(1,f(1)) \Rightarrow$ $y-y_1=m_{tan}(x-x_1)$

$$f'(x)=\left(\frac{x^2+5}{x+5}\right)' = \frac{(x^2+5)'(x+5)-(x^2+5)(x+5)'}{(x+5)^2}$$

$$f'(x)=\frac{2x(x+5)-(x^2+5)\cdot 1}{(x+5)^2}$$ (Do not simplify. Eval. at $x=1$)

$$f'(1)=\frac{2\cdot 1(1+5)-(1^2+5)\cdot 1}{(1+5)^2}$$

$$f'(1) = \frac{2\cdot 6 - 6}{6^2} = \frac{6}{6^2} = \frac{1}{6} = m_{tan}$$

$$\left( f(x) = \frac{x^2+5}{x+5} \right) \Rightarrow f(1) = \frac{1^2+5}{1+5} = \frac{6}{6} = 1 \quad (1,1)$$

Use the point-slope form:

$m_{tan} = \frac{1}{6}$

$(1,1)$

$$y - y_1 = m_{tan}(x - x_1)$$

$$y - 1 = \frac{1}{6}(x-1)$$

Find the equation of the $\underbrace{\text{normal line}}_{\text{perpendicular to the tangent line at } x=1.}$ to $f(x)$ at $x=1$.

$$m_{normal} = \frac{-1}{m_{tan}}$$

$$m_{normal} = \frac{-1}{\frac{1}{6}} = -6$$

eq. of the normal line:

$$y - 1 = -6(x-1)$$

Exp7) Find the coordinates of each point on the graph of $f(x)=\frac{1}{\sqrt{x}}(x+9)$ where the tangent line is horizontal.

Solution: Tangent line is horizontal means $m_{tan}=0$ at that point(s).

$$m_{tan}=f'(x)=0$$

$$f(x)=x^{-1/2}\cdot(x+9)=x^{1/2}+9\cdot x^{-1/2}$$

$$f'(x)=\frac{1}{2}\cdot x^{-1/2}+9\cdot\left(-\frac{1}{2}\right)\cdot x^{-3/2}=0$$

$$\frac{1}{2}\cdot x^{-3/2}\left(x^{1}-9\right)=0$$

$$\underbrace{\frac{1}{2\cdot x^{3/2}}}_{0}\cdot\underbrace{(x-9)}_{0}=0$$

zero property of multiplication:

$\frac{1}{2\cdot x^{3/2}} = 0$ or $x - 9 = 0$

no solution $\qquad$ $x = 9$

$x = 9 \Rightarrow f(9) = \frac{1}{\sqrt{9}}(9+9) = \frac{1}{3} \cdot 18 = 6$

Point $(9, 6)$ $\longrightarrow$ where $m_{tan} = 0$.

Exp8) Calculate $f'(x)$, $f''(x)$

when $f(x)=\dfrac{x-3}{1-x}$

Solution: $f(x) \to f'(x)$ (use quotient rule)

$$f'(x)=\frac{(x-3)'\cdot(1-x)-(x-3)(1-x)'}{(1-x)^2}$$

$$f'(x)=\frac{1\cdot(1-x)+(x-3)(+1)}{(1-x)^2}$$

$$=\frac{1-\cancel{x}+\cancel{x}-3}{(1-x)^2}=\frac{-2}{(1-x)^2}$$

$f'(x) \to f''(x)$

(may use the quotient rule but there's an easier way with chain rule later)

(for now, we must use the quotient rule)

$$f'(x) = \frac{-2}{(1-x)^2} = \frac{-2}{1-2x+x^2}$$

$$f''(x) = \frac{(-2)' \cdot (1-2x+x^2) + (+2) \cdot (1-2x+x^2)'}{[(1-x)^2]^2}$$

$$= \frac{0 + 2(-2+2x)}{(1-x)^4}$$

$$= \frac{-4+4x}{(1-x)^4} = \underbrace{\frac{-4(1-x)}{(1-x)^4} = \frac{-4}{(1-x)^3}}_{\text{factor the numerator and simplify}}$$

Exp9) Find the equation of the normal line to $f(x) = x^2 \cdot \tan x$ at $x = \pi/4$.

Solution: The normal line to $f(x)$ at $x=a$ is the line perpendicular to the tangent line at the point $(a, f(a))$.

Find $f'(x)\Big|_{x=\pi/4} = \underbrace{m_{tan}\Big|_{x=\pi/4}}_{\text{"slope of the tangent line at } x=\pi/4\text{"}} = f'\left(\frac{\pi}{4}\right)$

$f'(x) = (x^2 \cdot \tan x)'$ (use product rule)

$= 2x \cdot \tan x + x^2 \cdot \sec^2 x$

$f'(\pi/4) = 2 \cdot \frac{\pi}{4} \cdot \tan\left(\frac{\pi}{4}\right) + \left(\frac{\pi}{4}\right)^2 \cdot \sec^2\left(\frac{\pi}{4}\right)$

$f'\left(\frac{\pi}{4}\right) = \frac{2\pi}{4} \cdot 1 \quad + \frac{\pi^2}{16} \left(\frac{2}{\sqrt{2}}\right)^2$

$$f'\left(\frac{\pi}{4}\right) = \underset{(2)}{\frac{2\pi}{4}} + \cancel{\frac{\pi^2}{16}}\,4 \cdot \cancel{\frac{4}{2}} = \frac{4\pi}{8} + \frac{\pi^2}{8}$$

$$f'\left(\frac{\pi}{4}\right) = m_{tan}\Big|_{x=\pi/4} = \frac{4\pi+\pi^2}{8}$$

$$m_{normal} = \frac{-1}{m_{tan}} \Rightarrow m_{normal} = \frac{-1}{\frac{4\pi+\pi^2}{8}}$$

$$m_{normal} = \frac{-8}{4\pi+\pi^2}$$

point $(x_1, y_1) \Rightarrow \left(\pi/4, f(\pi/4)\right)$

$f(x) = x^2 \cdot \tan x$

$$\underbrace{f\left(\frac{\pi}{4}\right) = \left(\frac{\pi}{4}\right)^2 \cdot \tan\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16} \cdot 1}$$

Use point-slope form to write the eq. of the normal line to $f(x)$ at $x = \pi/4$:

$$y - y_1 = m_{normal}(x - x_1)$$

$$y - \frac{\pi^2}{16} = \frac{-8}{4\pi + \pi^2}\left(x - \frac{\pi}{4}\right)$$

Compare & contrast (3.1. vs. 3.2) methods

(Finding the equation of the tangent line by using the limit definition of derivative vs. rules of derivatives)

Find the slope of the tangent line to $f(x)=\frac{1}{x}$ at $x=4$.

By using the limit definition of derivative:

$$f'(4)=\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}=\lim_{x\to 4}\frac{\frac{1}{x}(4)-\frac{1}{4}(x)}{x-4}$$

$$=\lim_{x\to 4}\frac{\frac{4}{4x}-\frac{x}{4x}}{x-4}=\lim_{x\to 4}\frac{\frac{4-x}{4x}}{x-4}$$

$$=\lim_{x\to 4}\left(\frac{4-x}{4x}\cdot\frac{1}{x-4}\right)=\lim_{x\to 4}\left(\frac{-\cancel{(x-4)}^{1}}{4x\cdot\cancel{(x-4)}}\right)$$

$$= \lim_{x \to 4} \left( \frac{-1}{4x} \right) \overset{\text{"DSP"}}{=} \frac{-1}{4 \cdot 4} = \frac{-1}{16}$$ ✓

By using the derivative rules:

$$f'(4) = \left( \frac{1}{x} \right)' \Big|_{x=4} = \left( x^{-1} \right)' \Big|_{x=4}$$ ✓

$$= \left( -x^{-2} \right) \Big|_{x=4} = \left( -\frac{1}{x^2} \right) \Big|_{x=4} = \frac{-1}{16}$$

We should get the same answer!