Note Title 2/13/2020

3.5. Chain Rule

Intro: Find $f'(x)$ when $f(x) = \sin x$, $f'(x) = \cos x$

Find $g'(x)$ when $g(x) = \underbrace{\sin}_{\text{outside function}}\underbrace{(x^2-5)}_{\text{inside function}}$?

prime (Lagrange) Notation

$$[f(g(x))]' = \underbrace{f'(g(x))}_{\text{derivative of outside function w/ respect to } g(x)} \cdot \underbrace{g'(x)}_{\text{derivative of inside function w/ respect to x}}$$

Leibniz Notation

$$\frac{d}{dx}[f(g(x))] = \frac{df}{dg} \cdot \frac{dg}{dx}$$

CHAIN RULE $\left[f(g(x))\right]' = f'(g(x)) \cdot g'(x)$

Exp1) Differentiate $h(x) = e^{-3x}$

Solution: Identify inside & outside functions

outside f. $f(x) = e^{x} \Rightarrow f'(x) = e^{x}$, $f'(g(x)) = e^{-3x}$

inside f. $g(x) = -3x \Rightarrow g'(x) = -3$

By using the chain rule:

$$h'(x) = \left[f(g(x))\right]' = f'(g(x)) \cdot g'(x)$$

$$= e^{-3x} \cdot (-3)$$

$$= -3 \cdot e^{-3x}$$

Exp2) Differentiate $\ln(2-x^{2/3})$

outside f. $f(x) = \ln x$

inside f. $g(x) = 2-x^{2/3}$

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

$$f(x)=\ln x \Rightarrow f'(x) = \frac{1}{x}, \quad f'(g(x)) = \frac{1}{2-x^{2/3}}$$

$$g(x) = 2-x^{2/3} \Rightarrow g'(x) = 0 - \frac{2}{3} \cdot x^{\frac{2}{3}-1} = \frac{-2}{3} \cdot x^{-1/3}$$

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

$$\frac{1}{2-x^{2/3}} \cdot \left(\frac{-2}{3} \cdot x^{-1/3}\right)$$

Exp3) Find $f'(x)$ when $f(x) = \sin(3x^2+5x-7)$

$$f'(x) = \underbrace{\left[\cos(3x^2+5x-7)\right]}_{\text{derivative of the outside } f.} \cdot \underbrace{(6x+5)}_{\text{derivative of the inside } f.}$$

Extended Power Rule

$$\left[f^n(g(x))\right]' = n \cdot f^{n-1}(g(x)) \cdot f'(g(x)) \cdot g'(x)$$

Exp4) Calculate $\frac{d}{dx}\left(\underbrace{(3x^4-7x+5)^3}_{h(x)}\right)$

$$\frac{dh}{dx} = h'(x) = 3\cdot(3x^4-7x+5)^2 \cdot (3x^4-7x+5)'$$

$$h'(x) = \underbrace{3\cdot(3x^4-7x+5)^2}_{\text{derivative of the outside } f.} \cdot \underbrace{(12x^3-7)}_{\text{derivative of the inside } f.}$$

outside $f. \to x^3$

inside $f. \to (3x^4-7x+5)$

Exp5) Calculate $\frac{d}{dx}\Big(\underbrace{\cos(x^2)}_{①}+\underbrace{5\left(\frac{3}{x}+4\right)^6}_{②}\Big)$

① $\frac{d}{dx}(\cos(x^2))$ Recall: $\cos(x^2) \neq \cos^2(x)$

outside f. $f(x) = \cos x \Rightarrow f'(x) = -\sin x$, $f'(g(x)) = -\sin(x^2)$

inside f. $g(x) = x^2 \Rightarrow g'(x) = 2x$

$$\frac{d}{dx}(\cos(x^2)) = \underbrace{-\sin(x^2)}_{\text{derivative of outside f.}} \cdot \underbrace{2x}_{\text{derivative of inside f.}}$$

② $\frac{d}{dx}\left(5\cdot\left(\frac{3}{x}+4\right)^6\right) = \frac{d}{dx}\left(5\left(3\cdot x^{-1}+4\right)^6\right)$

outside f.

$f(x) = 5\cdot x^6 \Rightarrow f'(x) = 30x^5 \Rightarrow f'(g(x)) = 30(3\cdot x^{-1}+4)^5$

inside f.

$g(x) = 3\cdot x^{-1}+4 \Rightarrow g'(x) = -3x^{-2}$

$$\frac{d}{dx}\left(5\cdot\left(\frac{3}{x}+4\right)^6\right) = 30(3\cdot x^{-1}+4)^5\cdot(-3x^{-2})$$
$$= -90x^{-2}(3\cdot x^{-1}+4)^5$$

$$\frac{d}{dx}\left(\cos(x^2)+5\left(\frac{3}{x}+4\right)^6\right) =$$

put ①, ② together

$$-2x\cdot\sin(x^2) - 90x^{-2}(3\cdot x^{-1}+4)^5$$

Exp6) Differentiate $h(x)$ when $h(x)=\sqrt[4]{\frac{x}{1-3x}}$

(re-write $h(x)$) $h(x)=\left(\frac{x}{1-3x}\right)^{\frac{1}{4}}$

outside f. $f(x)= x^{1/4} \Rightarrow f'(x)=\frac{1}{4}\cdot x^{-3/4}$

$$f'(g(x)) = \frac{1}{4}\left(\frac{x}{1-3x}\right)^{-3/4}$$

$$g(x)=\frac{x}{1-3x} \Rightarrow g'(x)=\frac{1\cdot(1-3x)-x\cdot(-3)}{(1-3x)^2}$$

$$g'(x) = \frac{1-\cancel{3x}+\cancel{3x}}{(1-3x)^2} = \frac{1}{(1-3x)^2}$$

$$\frac{dh}{dx} = f'(g(x)) \cdot g'(x) = \boxed{\frac{1}{4} \cdot \left(\frac{x}{1-3x}\right)^{-3/4} \cdot \frac{1}{(1-3x)^2}}$$

Exp7) Differentiate $p(x) = \frac{\tan(7x)}{(1-4x)^5}$ $\rightarrow f$ $\rightarrow g$

Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

$$p'(x) = \frac{dp}{dx} = \frac{(\tan(7x))' \cdot (1-4x)^5 - \tan(7x) \left[(1-4x)^5\right]'}{\left[(1-4x)^5\right]^2}$$

$$p'(x) = \frac{7 \cdot \sec^2(7x)\,(1-4x)^5 - \tan(7x) \cdot 5 \cdot (1-4x)^4 \cdot (-4)}{(1-4x)^{10}}$$

Exp8) Find the x-coordinates of each point on the graph of $f(x) = x^2 \cdot (4x+5)^3$ where the tangent line is horizontal.

Solution:

$m_{tan} = 0 = f'(x)$

product rule $\quad f'(x) = 2x(4x+5)^3 + x^2 \cdot 3(4x+5)^2 \cdot 4 = 0$

factoring $\quad = 2x \cdot (4x+5)^2 \left[(4x+5)' + 6x\right] = 0$

zero property of multiplication $\quad = \underbrace{2x}_{0} \cdot \underbrace{(4x+5)^2}_{0} \cdot \underbrace{(10x+5)}_{0} = 0$

$2x = 0 \Rightarrow x = 0$

$(4x+5)^2 = 0 \Rightarrow 4x+5 = 0 \Rightarrow x = -5/4$

$(10x+5) = 0 \Rightarrow x = -1/2$

$x = 0, -5/4, -1/2$

Exp9) Find the x-coordinates of each point on the graph of $f(x)=\frac{\ln(\sqrt{x})}{x^2}$ where the tangent line is horizontal.

Solution:

Recall: $\ln(\sqrt{x}) = \ln(x^{1/2}) = \frac{1}{2} \cdot \ln x$

tangent line is horizontal means the slope of the tangent line is zero at that point.

$$m_{tan} = f'(x) = 0$$

By using properties of natural logs:

$$f(x) = \frac{\ln(x^{1/2})}{x^2} = \frac{\frac{1}{2} \cdot \ln x}{x^2} = \frac{\ln x}{2x^2} \begin{matrix} \to f \\ \to g \end{matrix}$$

use quotient rule: $f'(x) = \frac{(\ln x)' \cdot 2x^2 - \ln x \cdot (2x^2)'}{(2x^2)^2}$

$$f'(x)=\frac{\frac{1}{x}\cdot 2x^2-\ln x\cdot(4x)}{4x^4}=\frac{2x-(4x)\cdot\ln x}{4x^4}$$

$$=\frac{\overset{1}{\cancel{2x}}(1-2\cdot\ln x)}{\cancel{4x^4}\,2x^3}=\underbrace{\frac{1-2\cdot\ln x}{2x^3}=0}$$

$$1-2\cdot\ln x=0$$
$$1=2\cdot\ln x$$
$$\frac{1}{2}=\ln x$$
$$\ln e$$
$$x=e^{1/2}$$

By using the chain rule:

$$f(x)=\frac{\ln\sqrt{x}}{x^2}\begin{matrix}\to f\\ \to g\end{matrix}\qquad \left(\frac{f}{g}\right)'=\frac{f'\cdot g-f\cdot g'}{g^2}$$

$$f'(x)=\frac{(\ln\sqrt{x})'\cdot x^2-(\ln\sqrt{x})\cdot(x^2)'}{(x^2)^2}$$

Recall: $(\ln\sqrt{x})' = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x}$

outside f. $f(x) = \ln x \to f'(x) = \frac{1}{x} \to f'(g(x)) = \frac{1}{\sqrt{x}}$

inside f. $g(x) = \sqrt{x} \to g'(x) = \frac{1}{2} \cdot x^{-1/2} = \frac{1}{2\sqrt{x}}$

$$f'(x) = \frac{(\ln\sqrt{x})' \cdot x^2 - (\ln\sqrt{x}) \cdot (x^2)'}{(x^2)^2}$$

$$= \frac{\frac{1}{2x} \cdot x^2 - (\ln\sqrt{x}) \cdot (2x)}{x^4}$$

factor out like terms $= \dfrac{\frac{1}{2} \cdot x - (\ln\sqrt{x}) \cdot 2x}{x^4}$

$$= \frac{{}^{1}\cancel{x}\left(\frac{1}{2} - (\ln\sqrt{x}) \cdot 2\right)}{\cancel{x^4}\ x^3} = \frac{\frac{1}{2} - 2 \cdot \ln\sqrt{x}}{x^3}$$

$f'(x)=0$ (horizontal tangent line)

$$f'(x)=\frac{\frac{1}{2}-2\cdot\ln\sqrt{x}}{x^3}=\frac{0}{1}$$

$$\frac{1}{2}-2\cdot\ln\sqrt{x}=0$$

$$\frac{1}{2}-2\cdot\ln\sqrt{x}=0$$

$$\frac{1}{2}=2\cdot\ln\sqrt{x}$$

$$\frac{1}{4}=\ln\sqrt{x}\Rightarrow e^{1/4}=\sqrt{x}=x^{1/2}$$

e

$$(e^{1/4})^2=(x^{1/2})^2=x$$

$$e^{2/4}=e^{1/2}=x$$

Exp10) Evaluate $\frac{dh}{dx}$ when $h(x) = \sin(x \cdot \ln x)$

Solution:

$$h(x) = \sin(x \cdot \ln x)$$

outside f. $f(x) = \sin x \Rightarrow f'(x) = \cos x,\ f'(g(x)) = \cos(x \cdot \ln x)$

inside f. $g(x) = x \cdot \ln x \Rightarrow g'(x) = \underbrace{1 \cdot \ln x + x \cdot \frac{1}{x}}_{\text{product rule}} = \ln x + 1$

chain rule:

$$h'(x) = f'(g(x)) \cdot g'(x)$$
$$= \cos(x \cdot \ln x) \cdot (\ln x + 1)$$