3.6. Implicit Differentiation

Note Title 2/24/2020

$y = \sin x \rightarrow \frac{dy}{dx} = y' = \cos x$

chain rule $y = \sin(x^2 - 2x) \rightarrow \frac{dy}{dx} = \cos(x^2 - 2x) \cdot (2x - 2)$

$y = y(x)$ [y is defined as a function of x (implicitly)]

Procedure for Implicit Differentiation

(1) Differentiate both sides of an equation with respect to x. Use chain rule when differentiating terms that contain y.

(2) Solve the equation algebraically for $\frac{dy}{dx}$

$\frac{dy}{dx}$ "derivative of y with respect to x"

Exp1) Find the slope of the tangent line to the circle $x^2+y^2=10$ at $P(-1,3)$

Step1) Differentiate wrt x

$y \to y(x)$ {y is a function of x}

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(10)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(10)$$

$$2x + 2y\cdot\frac{dy}{dx} = 0$$

Step2) Solve for $\frac{dy}{dx}$

$$\frac{\cancel{2y}\cdot\frac{dy}{dx}}{\cancel{2y}} = \frac{-2x}{2y} \Rightarrow \frac{dy}{dx} = \frac{-x}{y} = m_{tan}$$

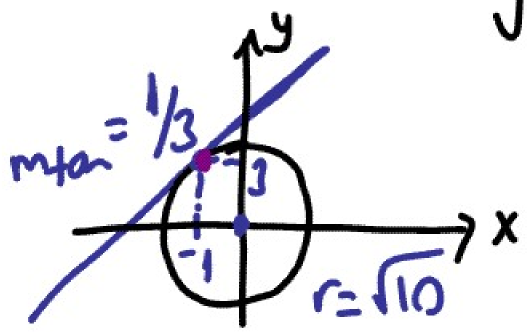
Recall:

$(x,y) \to (-1,3)$ $m_{tan}\Big|_{(-1,3)} = \frac{dy}{dx}\Big|_{(-1,3)} = \frac{-(-1)}{3} = \frac{1}{3}$

Eq. of a circle when center is at the origin:

$$x^2+y^2=r^2$$

$$x^2+y^2=10 \to r=\sqrt{10}$$



Exp2) Find $\frac{dy}{dx}$ for $\sin(x^2+y)=y^2(3x+1)$

Step 1) $$\frac{d}{dx}\left(\sin(x^2+y)\right) = \frac{d}{dx}\left(y^2\cdot(3x+1)\right)$$

$$\cos(x^2+y)\cdot\underbrace{[x^2+y]'}_{\text{same as } \frac{d}{dx}(x^2+y)} = 2y\cdot\frac{dy}{dx}\cdot(3x+1)+y^2\cdot 3$$

$$\cos(x^2+y)\cdot\left(2x+\frac{dy}{dx}\right) = 2y\cdot(3x+1)\cdot\frac{dy}{dx}+3y^2$$

Step 2) Solve for dy/dx (some algebra)

$$2x\cdot\cos(x^2+y)+\cos(x^2+y)\cdot\frac{dy}{dx}=2y\cdot(3x+1)\cdot\frac{dy}{dx}+3y^2$$

$$\frac{2x\cdot\cos(x^2+y)-3y^2}{(2y(3x+1)-\cos(x^2+y))}=\frac{\frac{dy}{dx}\cancel{(2y(3x+1)-\cos(x^2+y))}}{\cancel{(2y(3x+1)-\cos(x^2+y))}}$$

$$\frac{dy}{dx}=\frac{2x\cdot\cos(x^2+y)-3y^2}{2y(3x+1)-\cos(x^2+y)}$$

Exp3) Find $\frac{dy}{dx}$ for $\frac{1}{y}+\frac{1}{x}=1$.

Re-write first:

$$y^{-1}+x^{-1}=1$$

Step1) $\frac{d}{dx}(y^{-1}) + \frac{d}{dx}(x^{-1}) = \frac{d}{dx}(1)$

$$-1 \cdot y^{-2} \cdot \frac{dy}{dx} - x^{-2} = 0$$

Step2) $\frac{-x^{-2}}{y^{-2}} = \frac{\cancel{y^{-2}} \cdot \frac{dy}{dx}}{\cancel{y^{-2}}}$

$$\frac{dy}{dx} = -\frac{x^{-2}}{y^{-2}} = \frac{-y^2}{x^2}$$

Recall: $x^{-2} = \frac{1}{x^2}$

Exp4) Find $\frac{d^2y}{dx^2}$ when $x^2+y^2=10$

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(10)$$

$$2x + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-x}{y}\right)$$

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d(-x)}{dx} \cdot y + (+x) \cdot \frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-1 \cdot y + x \cdot \frac{dy}{dx}}{y^2}$$

$$\left(\text{Recall: } \frac{dy}{dx} = \frac{-x}{y}\right)$$

$$= \frac{-y + x \cdot \left(\frac{-x}{y}\right)}{y^2} = \frac{\frac{-y}{1}\frac{(y)}{(y)} - \frac{x^2}{y}}{y^2}$$

Recall: $x^2 + y^2 = 10$

$$= \frac{\frac{-y^2 - x^2}{y}}{\frac{y^2}{1}} = \frac{-(y^2 + x^2)}{y} \cdot \frac{1}{y^2} = \frac{-10}{y^3}$$

Exp5) Find an equation of the tangent line to the graph of $\sin(x-y)=x\cdot y$ at $(0,\pi)$

Step1) $\cos(x-y)\cdot\frac{d}{dx}(x-y) = \frac{dx}{dx}\cdot y + x\cdot\frac{dy}{dx}$

Recall: $m_{tan} = \frac{dy}{dx}$

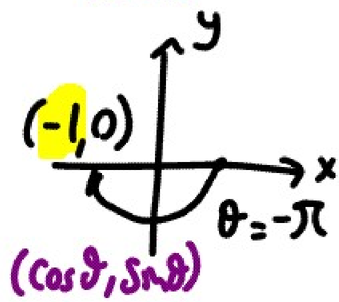
$$\cos(x-y)\cdot\left(1-\frac{dy}{dx}\right) = y + x\cdot\frac{dy}{dx}$$

substitute $(0,\pi)$ as (x,y)

$$\cos(0-\pi)\cdot\left(1-\frac{dy}{dx}\right) = \pi + \cancelto{0}{0\cdot\frac{dy}{dx}}$$

$$\underbrace{\cos(-\pi)}_{-1}\cdot\left(1-\frac{dy}{dx}\right) = \pi$$

Recall:



$$-1\left(1-\frac{dy}{dx}\right) = \pi$$

$$-1+\frac{dy}{dx} = \pi \Rightarrow \frac{dy}{dx} = \pi+1$$

$$\left.\frac{dy}{dx}\right|_{(0,\pi)} = m_{tan}\Big|_{(0,\pi)} = \pi+1$$

$(x,y) \rightarrow (0,\pi)$

$m \rightarrow \pi+1$

eq. of the tangent line:

$$y-\pi = (\pi+1)\cdot(x-0)$$

eq. of the normal line:

$$m_{normal} = \frac{-1}{m_{tan}} = \frac{-1}{\pi+1}$$

$$y-\pi = \left(\frac{-1}{\pi+1}\right)(x-0)$$

Logarithmic Differentiation
(both base & exponent are functions "f^f")

Exp6) Find $\frac{dy}{dx}$ when $y=x^x$

$$y=x^x$$

$$\ln(y)=\ln(x^x)$$ [property of logs]

$$\ln(y)=x\cdot\ln(x)$$

$$\frac{d}{dx}(\ln(y))=\frac{d}{dx}(x\cdot\ln(x))$$ [differentiate]

$$\cancel{y}\cdot\frac{1}{\cancel{y}}\cdot\frac{dy}{dx}=\left(1\cdot\ln x+x\cdot\frac{1}{x}\right)\cdot y$$ [solve for $\frac{dy}{dx}$]

$$\frac{dy}{dx}=(\ln x+1)\cdot y$$ [sub x^x for y]

$$\frac{dy}{dx}=(\ln x+1)\cdot x^x$$

Exp7) Find $\frac{dy}{dx}$ when $y=(1+\sin(2x))^{x^2}$

$$\ln(y)=\ln\left(1+\sin(2x)\right)^{x^2}$$

$$\ln(y)=\underbrace{x^2\cdot\ln\left(1+\sin(2x)\right)}_{\text{product rule}}$$

$$\frac{1}{y}\cdot\frac{dy}{dx}=2x\cdot\ln\left(1+\sin(2x)\right)+x^2\cdot\frac{d}{dx}\left(\ln(1+\sin(2x))\right)$$

$$y\cdot\frac{1}{y}\cdot\frac{dy}{dx}=\left[2x\cdot\ln\left(1+\sin(2x)\right)+x^2\cdot\frac{2\cdot\cos(2x)}{1+\sin(2x)}\right]\cdot y$$

$$\frac{dy}{dx}=\left[2x\cdot\ln\left(1+\sin(2x)\right)+\frac{2x^2\cdot\cos 2x}{1+\sin 2x}\right]\cdot\underbrace{\left(1+\sin(2x)\right)^{x^2}}$$

Recall:

$y=(1+\sin(2x))^{x^2}$

Exp8) Find the slope of the tangent line to the graph of $x^3+y^3-\frac{9}{2}x\cdot y=0$ at $(2,1)$.

$$m_{tan}\Big|_{(2,1)} = \frac{dy}{dx}\Big|_{(2,1)}$$

$$3x^2+3y^2\cdot\frac{dy}{dx}-\left(\frac{9}{2}\cdot 1\cdot y+\frac{9}{2}\cdot x\cdot\frac{dy}{dx}\right)=0$$

sub $(2,1)$ for (x,y) to make Algebra easier

$$3\cdot 2^2+3\cdot 1^2\cdot\frac{dy}{dx}-\left(\frac{9}{2}\cdot 1+\frac{9}{2}\cdot 2\cdot\frac{dy}{dx}\right)=0$$

$$12+3\cdot\frac{dy}{dx}-\left(\frac{9}{2}+\frac{9\cdot dy}{dx}\right)=0$$

solve for dy/dx

$$12+3\cdot\frac{dy}{dx}-\frac{9}{2}-9\cdot\frac{dy}{dx}=0$$

$$-6 \cdot \frac{dy}{dx} + \frac{15}{2} = 0$$

$$\frac{-6 \cdot \frac{dy}{dx}}{-6} = \frac{\frac{-15}{2}}{-6}$$

$$\frac{dy}{dx} = \frac{5}{4}$$