Note Title

3.7. Related Rates and Applications

2/27/2020

We will relate the rates of change of different variables with respect to time.

Procedure for solving related rates prb.

① Draw a figure, assign variables to quantities that vary (What's not changing (a constant) vs. what's changing (variable))

② Find a formula or an equation that relates the variables

③ Differentiate the equation (usually implicitly w/ respect to time)

④ Substitute specific values and solve algebraically for any required rate (use correct units)

Steps 1 & 2 involve reading & interpreting the prb.
Step 3 is implicit differentiation / Step 4 Algebra

Expl) A spherical balloon is filled w/ gas. When $r=2ft$, the radius is increasing at the rate of $\frac{1}{6}\frac{ft}{min}$. How fast is the volume changing at this time?

$\left(V=\frac{4}{3}\pi r^3\right)$

Solution:

① air

balloon

$r \to$ radius

$V \to$ volume

Given: $r=2ft.$

$\frac{dr}{dt}=+\frac{1}{6}\frac{ft}{min}$ (rate of change of radius wrt time)

Asked: $\frac{dV}{dt}=?$ (how is the Volume changing wrt time?)

② Formula: $V=\frac{4}{3}\pi r^3$

③ Implicit Diff. wrt (w/respect to) time:

$(V=V(t))\quad V=\underbrace{\frac{4}{3}}_{\text{constant}}\pi r^3\quad (r=r(t))$

$$\frac{dV}{dt} = \frac{4}{3} \cdot \pi \cdot 3 \cdot r^2 \cdot \frac{dr}{dt}$$

Given:

(4) At the moment when $r = 2\,ft$, $\frac{dr}{dt} = \frac{1}{6}\,ft/min.$

substitute these specific values in the eq. step (3)

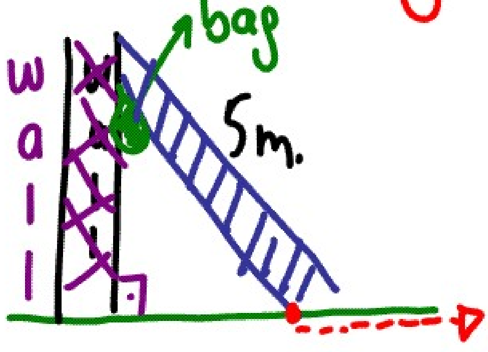
$$\frac{dV}{dt} = \frac{4}{3} \cdot \pi \cdot 3 \cdot r^2 \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = \frac{\cancel{4}^{2}}{\cancel{3}} \cdot \pi \cdot \cancel{3} \cdot 2^2 \cdot \frac{1}{\cancel{6}_{3}} = \frac{8\pi}{3}\,\frac{ft^3}{min.}$$

Volume is increasing at a rate of $\frac{8\pi}{3}\,\frac{ft^3}{min}$.*

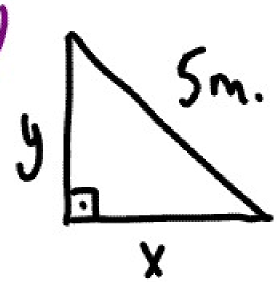
Units* how do we measure volume? in this
how do we measure time? prb.

Exp2) Leaning Ladder Prb.



when the foot of the ladder is 4m away from the wall and the foot is moving away at the rate of 2 m/sec. how fast is the bag descending?

(1)



Given:
$x \to$ distance between foot of the ladder and the wall

$y \to$ distance between the bag and the ground

Asked: when $x = 4m.$, $\frac{dx}{dt} = 2m/s.$ $\frac{dy}{dt} = ?$

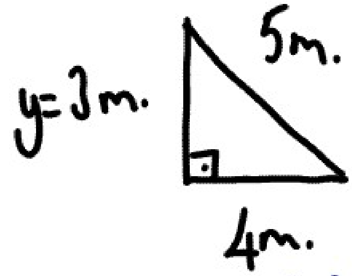
(2) $$x^2 + y^2 = 5^2$$

(Is y increasing wrt time? NO!)

(3) $$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

Given: Asked:

4) when $x = 4m.$, $\frac{dx}{dt} = 2m/s.$; $\frac{dy}{dt} = ?$



3-4-5 Special Right Triangle

$y^2 + 4^2 = 5^2$

$y^2 = 25 - 16 = 9$

$y = 3m.$

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

Substitute given (specific) values:

$$2 \cdot 4 \cdot 2 + 2 \cdot 3 \cdot \frac{dy}{dt} = 0$$

$$16 + 6 \cdot \frac{dy}{dt} = 0$$

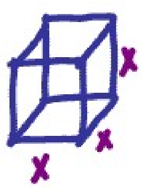
$$\frac{dy}{dt} = \frac{-16}{6} = \frac{-8}{3} \text{ m/sec.}$$

The bag is $\underbrace{\text{descending}}_{\text{already implies a neg. change}}$ at a rate of $\frac{8}{3} \frac{m}{sec}$.

Exp3) Fall 2019 Midterm #2 Q

The edges of a cube increase at a rate of 2cm/sec. How fast is the volume changing when the length of each edge is 50 cm? You must include correct units.

(1) Cube: all edges (x) are the same.



Given: when $x = 50$ cm, $\frac{dx}{dt} = 2$ cm/sec.

Asked: $\frac{dV}{dt} = ?$

(2) Relate the variables (edges and volume) by a formula: $V(x) = x^3$

(3) Relate the rate of change of the variables wrt time by using implicit differentiation.

$$V = x^3$$

$$\frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt}$$

4. When $x = 50$ cm. and $\frac{dx}{dt} = 2$ cm/sec, $\frac{dV}{dt} = ?$

$$\frac{dV}{dt} = 3 \cdot 50^2 \cdot 2 = 3 \cdot 50 \cdot 50 \cdot 2$$
$$= 150 \cdot 100$$
$$= 15{,}000$$

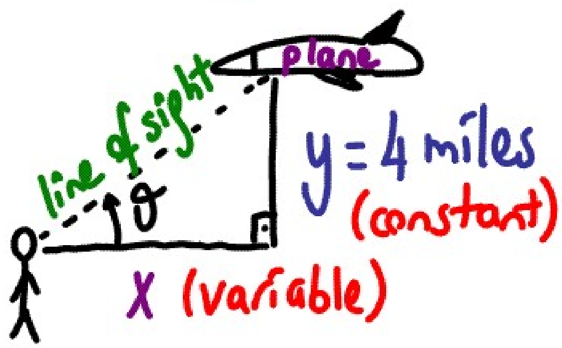
The volume is increasing at a rate of 15,000 cm^3/sec. at that specific time.

Exp4) Angle of Elevation

A plane is flying at a constant speed of 400 $\frac{\text{miles}}{\text{hr}}$ at a constant altitude of 4 miles. At what rate is the angle of elevation of my line of sight changing wrt time when the horizontal distance between the approaching plane and my location is exactly 3 miles?

Solution

① Draw a figure & identify given/asked



$x \to$ horizontal distance between plane and the person

$y \to$ constant altitude

$\theta \to$ angle of elevation

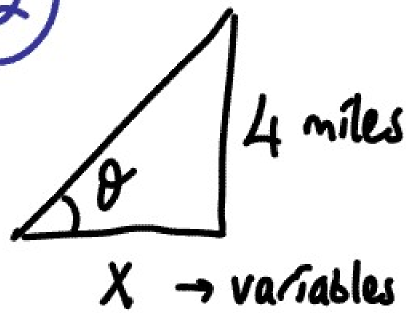
* height of the person is insignificant compared to the vertical distance $y = 4$ miles

Given: when $x = 3$ miles, $\frac{dx}{dt} = -400 \frac{\text{miles}}{\text{hr.}}$

(negative since x is decreasing by time)

Asked: @ that point in time, how's θ changing? $\left(\frac{d\theta}{dt} = ?\right)$

(2)



SOH CAH TOA

$$\tan\theta = \frac{\text{Opp}}{\text{Adj.}} = \frac{4}{x}$$

(3) $$\tan\theta = \frac{4}{x} \Rightarrow \tan\theta = 4\cdot x^{-1}$$

differentiate both sides wrt time

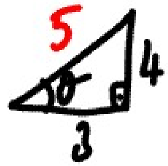
$$\sec^2\theta \cdot \frac{d\theta}{dt} = 4\cdot(-1)\cdot x^{-2}\cdot\frac{dx}{dt}$$

(both x, θ changes wrt time)

(4) when $x = 3$ miles, $\frac{dx}{dt} = -400\ \frac{\text{miles}}{\text{hr.}}$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = -4\cdot x^{-2}\cdot\frac{dx}{dt}$$

Find $\sec\theta$ first.



$$\sec\theta = \frac{1}{\cos\theta} = \frac{5}{3}$$

$$\left(\frac{5}{3}\right)^2 \cdot \frac{d\theta}{dt} = -4 \cdot 3^{-2} \cdot (-400)$$

$$\frac{9}{25} \cdot \frac{25}{9} \cdot \frac{d\theta}{dt} = +4 \cdot \frac{1}{9} \cdot (+400) \cdot \frac{9}{25}$$

$$\frac{d\theta}{dt} = 4 \cdot \frac{1}{\cancel{9}} \cdot \overset{16}{\cancel{400}} \cdot \frac{\cancel{9}}{\cancel{25}}$$

$$\frac{d\theta}{dt} = 64 \ \frac{rad}{hr}$$ (in Trig, angles are in radians)

The angle of elevation is increasing at a rate of 64 rad/hr.