Note Title 2/17/2020

Midterm#1 Review

Exp1) Solve the inequality in interval notation:

$$\frac{2x-4}{x+3} > 0$$

Solution: Set both the numerator, denominator to 0.

$$2x-4=0 \qquad x+3=0$$
$$x=2 \qquad x=-3$$

$\left.\right\}$ $x=-3$, $x=2$ cutpoints

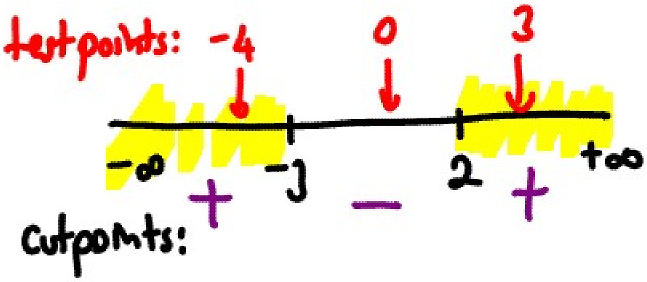
testpoints:
$x=-4, 0, 3$

Let $f(x)=\frac{2x-4}{x+3}$

$$f(-4)=\frac{-8-4}{-4+3}=\frac{\ominus}{\ominus} > 0 \quad \checkmark$$

$$f(0)=\frac{0-4}{0+3}=\frac{\ominus}{\oplus} < 0 \quad \times$$

$$f(3)=\frac{2\cdot 3-4}{3+3}=\frac{\oplus}{\oplus} > 0 \quad \checkmark$$



Solution:
$(-\infty,-3)$, $(2,\infty)$
$(-\infty,-3)\cup(2,\infty)$
$(-\infty,-3)$ OR $(2,\infty)$
All acceptable!

Exp2) Solve the equation:

a) $\dfrac{e^{x^2}}{e^{x+6}} = 1 \Rightarrow \cancel{e}^{x^2} = \cancel{e}^{x+6}$

$x^2 = x+6$

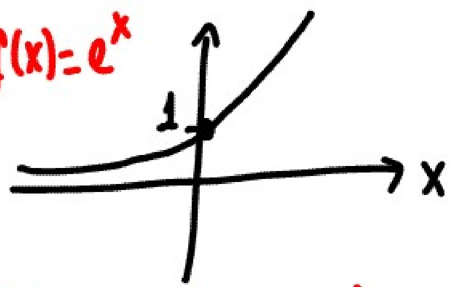
$x^2 - x - 6 = 0$

$-3 \quad 2$

$(x-3)\cdot(x+2) = 0$

$x = 3, \quad x = -2$

$f(x) = e^x$



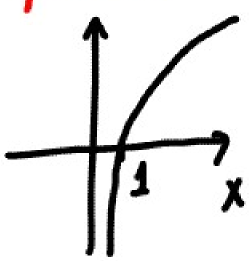
Domain: $(-\infty, \infty)$

b) $\ln(x) + \ln(2x+1) = 0$

$\boxed{x = \frac{1}{2}}$

$f(x) = \ln x$

$\ln(x\cdot(2x+1)) = 0$



$\ln(2x^2+x) = 0 \Rightarrow$

e

$2x^2 + x = e^0 = 1$

$2x^2 + x - 1 = 0$

Domain: $(0, \infty)$

$(2x-1)(x+1) = 0$

$x = \frac{1}{2}, \quad x = -1$ not in domain!

Therefore, the only solution is $\boxed{x=\frac{1}{2}}$

Exp3) Evaluate limits or show DNE.

a) $\lim\limits_{x\to 1}\left(\frac{5-\sqrt{32-7x}}{x-1}\right) \overset{\text{"DSP"}}{=} \frac{5-\sqrt{32-7\cdot 1}}{1-1} = \text{"}\frac{0}{0}\text{"}$

$\lim\limits_{x\to 1}\left(\frac{5-\sqrt{32-7x}}{x-1}\cdot\frac{5+\sqrt{32-7x}}{5+\sqrt{32-7x}}\right)$ Recall: $(a-b)(a+b)=a^2-b^2$

$\lim\limits_{x\to 1}\left(\frac{5^2-(\sqrt{32-7x})^2}{(x-1)\cdot(5+\sqrt{32-7x})}\right)$

$\lim\limits_{x\to 1}\left(\frac{25-(32-7x)}{(x-1)(5+\sqrt{32-7x})}\right)$

$-7+7x$

$\lim\limits_{x\to 1}\left(\frac{25-32+7x}{(x-1)(5+\sqrt{32-7x})}\right) = \lim\limits_{x\to 1}\frac{7(\cancel{-1+x})}{\cancel{(x-1)}(5+\sqrt{32-7x})}$ $x-1$

$\lim\limits_{x\to 1}\frac{7}{5+\sqrt{32-7x}} \overset{\text{"DSP"}}{=} \frac{7}{5+\sqrt{32-7}}$

$= \frac{7}{5+\sqrt{25}} = \boxed{\frac{7}{10}}$

b) $\lim_{x\to 3} \dfrac{x^4-81}{2x^2-5x-3}$

Recall:

$(a^2-b^2)=(a-b)(a+b)$

$$\lim_{x\to 3} \frac{(x^2)^{\cancel{2}}-9^{\cancel{2}}}{2x^2-5x-3} = \lim_{x\to 3} \frac{(x^2-9)(x^2+9)}{2x^2-5x-3}$$

$2x \quad 1$

$x \quad -3$

$-6x+x=-5x$ ✓

$$\lim_{x\to 3} \frac{\cancel{(x-3)}(x+3)(x^2+9)}{(2x+1)\cancel{(x-3)}}$$

$$\lim_{x\to 3} \frac{(x+3)(x^2+9)}{(2x+1)} \overset{\text{"DSP"}}{=} \frac{6\cdot 18}{7} = \frac{108}{7}$$

c) $\lim_{x\to 0} \dfrac{\sin(3x)}{\sin(2x)}$

$\left[\lim_{x\to 0} \dfrac{\sin(kx)}{kx} = 1\right]$

$$\lim_{x\to 0} \left(\frac{\sin(3x)}{\sin(2x)} \cdot \frac{3x}{3x} \cdot \frac{2x}{2x}\right)$$

$$\lim_{x\to 0} \left(\underbrace{\frac{\sin(3x)}{3x}}_{1} \cdot 3x \cdot \underbrace{\frac{2x}{\sin(2x)}}_{1} \cdot \frac{1}{2x}\right)$$

$$\lim_{x\to 0} \left(\frac{3x}{2x}\right) = \frac{3}{2}$$

4 Calculate $f'(x)$, don't simplify

a) $f(x) = \cos(6x \cdot e^{-2x})$

$$f'(x) = \underbrace{-\sin(6x \cdot e^{-2x})}_{\text{derivative of outside}} \cdot \underbrace{[6x \cdot e^{-2x}]'}_{\text{derivative of inside}}$$

$$= -\sin(6x \cdot e^{-2x}) \cdot (6 \cdot e^{-2x} + 6x \cdot e^{-2x} \cdot -2)$$

b) $g(x) = x^2 \cdot e^{-x} + (\ln x)^2$

$$g'(x) = 2x \cdot e^{-x} + x^2 \cdot (-1 \cdot e^{-x}) + 2 \cdot (\ln x)^1 \cdot (\ln x)'$$

$$= 2x \cdot e^{-x} - x^2 \cdot e^{-x} + 2 \cdot \underbrace{\ln x \cdot \frac{1}{x}}_{\ln x \cdot \frac{1}{x} \neq \ln 1}$$

Be careful!

Fall 2019 my midterm#1 Q (part b)

Exp5) a) Calculate $f'(x)$ by using the derivative rules

b) Use the limit definition of derivative to evaluate $f'(x)$

$$f(x) = \sqrt{7x}$$

a) $f(x) = (7x)^{1/2} \Rightarrow f'(x) = \frac{1}{2} \cdot (7x)^{-1/2} \cdot 7$

$$\boxed{f'(x) = \frac{7}{2} \cdot \frac{1}{\sqrt{7x}}}$$

b) $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f(x) = \sqrt{7x}$

$f(x+h) = \sqrt{7(x+h)}$

$$\lim_{h \to 0} \left(\frac{\sqrt{7(x+h)} - \sqrt{7x}}{h} \cdot \frac{\sqrt{7(x+h)} + \sqrt{7x}}{\sqrt{7(x+h)} + \sqrt{7x}} \right)$$

$$\lim_{h \to 0} \frac{7(x+h) - 7x}{h(\sqrt{7(x+h)} + \sqrt{7x})} = \lim_{h \to 0} \frac{7\cancel{h}}{\cancel{h}(\sqrt{7(x+h)} + \sqrt{7x})}$$

$$\lim_{h \to 0} \frac{7}{\sqrt{7(x+h)}+\sqrt{7x}} \overset{\text{"DSP"}}{=} \frac{7}{\sqrt{7(x+0)}+\sqrt{7x}} = \frac{7}{2\sqrt{7x}}$$

Exp6) Find the equation of the normal line to $f(x)=\frac{1}{x+3}$ at $x=2$. You must use the limit def. of derivative and proper notation. Any form of an equation is acceptable.

If there's no point (general form)

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$x=a$ (at a point)

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$x=2$; $f(x)=\frac{1}{x+3}$, $f(2)=\frac{1}{5}$ use $\lim\limits_{x \to 2} \frac{f(x)-f(2)}{x-2}$

$$\lim_{x \to 2} \frac{\frac{1}{x+3}{\scriptstyle(5)} - \frac{1}{5}{\scriptstyle(x+3)}}{x-2} = \lim_{x \to 2} \frac{\frac{5}{5(x+3)} - \frac{(x+3)}{5(x+3)}}{x-2}$$

$$\lim_{x\to 2} \frac{\frac{5-(x+3)}{5(x+3)}}{x-2} = \lim_{x\to 2} \frac{\frac{5-x-3}{5(x+3)}}{x-2}$$

Recall:
$2-x=-(x-2)$

$$\lim_{x\to 2} \frac{\frac{\cancel{2-x}\ -1}{5(x+3)}}{\cancel{x-2}} = \lim_{x\to 2} \frac{-1}{5(x+3)} \overset{\text{"DSP"}}{=} \frac{-1}{5\cdot 5} = \frac{-1}{25}$$

$$m_{tan}\Big|_{x=2} = \frac{-1}{25} \qquad m_{normal} = \frac{-1}{m_{tan}}$$

$$m_{normal} = \frac{-1}{-1/25} = 25$$

(x_1, y_1)

$$y - y_1 = m_{normal}(x - x_1) \qquad \left(2, \frac{1}{5}\right)$$

Eq. of the normal line: $y - \frac{1}{5} = 25(x-2)$

Eq. of the tangent line: $y - \frac{1}{5} = \frac{-1}{25}(x-2)$

Exp 8a) Determine if $g(x)$ is continuous at each value:

$$g(x)=\begin{cases} x+e^x, & x<0 & ① \\ x^2, & x\geq 0 & ② \end{cases}$$

For $g(x)$ to be continuous at $x=0$ (transition point)

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x) = g(0)$$

LL → ← RL

① 0 ②

$$\lim_{x\to 0^-}(x+e^x) \overset{\text{"OSP"}}{=} 0+e^0 = 1$$

$$\lim_{x\to 0^+}(x^2) \overset{\text{"OSP"}}{=} 0^2 = 0$$

$1 \neq 0$

$\lim_{x\to 0} f(x)$ DNE

$$f(0) = 0^2 = 0$$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x) \neq f(0)$

$f(x)$ is discontinuous at $x=0$.

Exp 8b) Calculate $\lim_{x \to 0} g(x)$ or show it DNE.

$$g(x) = \begin{cases} x + e^x, & x < 0 \quad ① \\ x^2, & x \geq 0 \quad ② \end{cases}$$

Solution: Unlike continuity; we have 2 conditions to check (left-limit and right-limit)

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (x + e^x) \overset{\text{"DSP"}}{=} 0 + e^0 = 1$$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x^2) \overset{\text{"DSP"}}{=} 0^2 = 0$$

Since $\lim_{x \to 0^-} g(x) \neq \lim_{x \to 0^+} g(x)$

$\lim_{x \to 0} g(x)$ DNE.

Exp9) Determine all values of A and B so that $f(x)$ is continuous for all values of x.

$$f(x) = \begin{cases} Ax - B, & \text{if } x \le -1 \\ 2x^2 + 3Ax + B, & \text{if } -1 < x \le 1 \\ 4, & \text{if } x > 1 \end{cases}$$

transition points: $x = -1, 1$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$$ (3 conditions of continuity at a point)

$$\lim_{x\to -1^-} (Ax - B) \overset{\text{"DSP"}}{=} A(-1) - B = -A - B$$

$$\lim_{x\to -1^+} (2x^2 + 3Ax + B) \overset{\text{"DSP"}}{=} 2(-1)^2 + 3A(-1) + B$$

$$= 2 - 3A + B$$

$$f(-1) = A(-1) - B = -A - B$$

$$-A - B = 2 - 3A + B \Rightarrow 2A - 2B = 2 \Rightarrow \boxed{A - B = 1}$$

Now, let's check the continuity at $x=1$
"DSP"

$$\lim_{x \to 1^-} (2x^2+3Ax+B) = 2+3A+B$$

$$\lim_{x \to 1^+} 4 = 4$$

$$\underbrace{f(1) = 2\cdot 1^2+3A\cdot(1)+B = 2+3A+B}$$

$$2+3A+B=4 \Rightarrow \boxed{3A+B=2}$$

Solve: $A-B=1$

$$\begin{array}{r} 3A+B=2 \\ + \quad\quad\quad \\ \hline \end{array}$$

$$4A=3 \Rightarrow \boxed{A=3/4}$$

$$A-B=1 \Rightarrow \frac{3}{4}-B=1 \Rightarrow \boxed{B=-\frac{1}{4}}$$

Exp10) Find an equation of the line tangent to $f(x)=x^2+\sin\left(\frac{\pi}{2}x\right)$ at $x=-1$.

Solution:

$$m_{tan}\Big|_{x=-1}=f'(-1)$$

$$f'(x)=2x+\cos\left(\frac{\pi}{2}x\right)\cdot\left(\frac{\pi}{2}\right)$$

$$f'(-1)=-2+\cos\left(\frac{-\pi}{2}\right)\cdot\left(\frac{\pi}{2}\right)$$

Recall:

$$\cos\left(\frac{-\pi}{2}\right)=0$$

$$f'(-1)=-2+0\cdot\left(\frac{\pi}{2}\right)=-2$$

Equation of the tangent line to $f(x)$ at $x=-1$:

$$y-y_1=m_{tan}(x-x_1)$$

$$y-0=-2(x-(-1))$$

$$y=-2(x+1)$$

Recall:

$$f(-1)=(-1)^2+\sin\left(\frac{\pi}{2}\cdot(-1)\right)$$

$$f(-1)=1+\sin\left(\frac{-\pi}{2}\right)=0$$

Exp 11) Calculate $f'(x)$. Do not simplify.

$$f(x)=\frac{\sqrt{x}-2}{e^x+\ln 8}$$

Solution: Use quotient rule:

$$f'(x)=\frac{(\sqrt{x}-2)'\cdot(e^x+\ln 8)-(\sqrt{x}-2)\cdot(e^x+\ln 8)'}{(e^x+\ln 8)^2}$$

$$=\frac{\frac{1}{2\sqrt{x}}\cdot(e^x+\ln 8)-(\sqrt{x}-2)\cdot(e^x+0)}{(e^x+\ln 8)^2}$$