Note Title

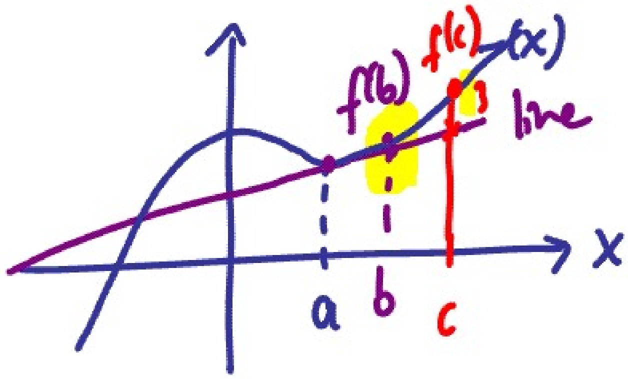
3.8. Linearization and Differentials

3/2/2020

Linearization → use tangent line function

→ approximate a value that's very close to a known, easy to compute value.

E.g: Approximate $\underbrace{\sqrt{4.01}}_{\text{very close to } \sqrt{4}}$ by using $\underbrace{\sqrt{4}=2}_{\text{known value}}$



As the # gets closer to "a", approximation gets better on the tangent line

Eq. of a tangent line at $x=a$

$$y - y_1 = m_{tan}(x - x_1)$$

substitute: $m_{tan}\big|_{x=a} = f'(a)$, $(a, f(a)) \rightarrow (x_1, y_1)$

$$y - f(a) = f'(a)(x-a)$$

$y = f(a) + f'(a)(x-a)$

$L(x)$ [linearization of $f(x)$ at $x=a$]

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

$a \to$ known, easy to compute value

$x \to$ unknown x-value

(x & a should be very close)

$f(x) \to$ function

Exp1) Use linear approximation to estimate the value of $\sqrt{35}$

$L(x) = f(a) + f'(a)(x-a)$

$a = 36$ (just the #, NOT $\sqrt{36}$)

$f(x) = \sqrt{x} \Rightarrow f(a) = f(36) = \sqrt{36} = 6$

$$L(x) \to L(35)$$

$$f(x) = \sqrt{x} = x^{1/2} \Rightarrow f'(x) = \frac{1}{2} \cdot x^{-1/2}$$

$$\Rightarrow f'(36) = \frac{1}{2} \cdot (36)^{-1/2}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{36}} = \frac{1}{12}$$

$$L(x) = f(a) + f'(a)(x-a)$$

$$L(35) = f(36) + f'(36)(35-36)$$

$$= 6 + \frac{1}{12} \cdot (-1) = 6 - \frac{1}{12} = \frac{71}{12}$$

Exp2) Use linear approximation to estimate:

$$(16.01)^{3/2} + 2 \cdot \sqrt{16.01}$$

$$L(x) = f(a) + f'(a)(x-a)$$

$a \to$ known, easy to compute #, close to 16.01 (16)

$f(x) = x^{3/2} + 2\cdot\sqrt{x} \to f'(x) = \frac{3}{2}\cdot x^{1/2} + 2\cdot\frac{1}{2}\cdot x^{-1/2}$

$L(x) = L(16.01)$

$f(a) = f(16) = 16^{3/2} + 2\sqrt{16}$

Recall:

$= (4^2)^{3/2} + 2\cdot\sqrt{16}$ $(x^m)^n = x^{m\cdot n}$

$= (4^{\cancel{2}\cdot\frac{3}{\cancel{2}}}) + 2\cdot 4$

$= 64 + 8 = 72$

$f'(x) = \frac{3}{2}\cdot x^{1/2} + \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot x^{-1/2}$

$f'(16) = \frac{3}{2}\cdot(16)^{1/2} + (16)^{-1/2}$

$= \frac{3}{2}\cdot 4 + \frac{1}{\sqrt{16}} = \frac{6}{1_{(4)}} + \frac{1}{4} = \frac{25}{4}$

$$L(x) = f(a) + f'(a)(x-a)$$

$$\underbrace{L(x) = 72 + \frac{25}{4}(x-16)}$$

$$L(16.01) = 72 + \frac{25}{4}\left(\underbrace{16.01 - 16}_{0.01}\right)$$

$$= 72 + \frac{\cancel{25}^{1}}{4} \cdot \frac{1}{\cancel{100}_{4}} = 72 + \frac{1}{16} = 72\frac{1}{16}$$

Exp3) Use linear approximation to estimate the value of $\cos\left(\frac{\pi}{2} + 0.01\right)$

$$L(x) = f(a) + f'(a)(x-a)$$

$f(x) = \cos(x)$

$a \to$ known value $\left(\frac{\pi}{2}\right)$

$x \to \frac{\pi}{2} + 0.01$

Note: we expect the approx. to be $(-)$ $\cos\left(\frac{\pi}{2}+0.01\right)$ is in the 2nd Q.

$$f(a) = f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$f'(x) = -\sin(x)$$

$$f'(a) = f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$$

$$L(x) = f(a) + f'(a)(x-a)$$

$$L\left(\frac{\pi}{2} + 0.01\right) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2} + 0.01 - \frac{\pi}{2}\right)$$

$$= 0 + (-1)(0.01) = -0.01$$

With calc: (in radians) $\cos\left(\frac{\pi}{2} + 0.01\right) \approx -0.00999983$

≈ -0.010

Differentials

$y = f(x)$

Differentiate both sides wrt x

$$dx \cdot \underbrace{\frac{dy}{dx}}_{\text{Leibniz Notation}} = \underbrace{f'(x)}_{\text{Lagrange Notation}} \cdot dx$$

(although $\frac{dy}{dx}$ is not division)

$$\boxed{\underbrace{dy}_{\text{differential of } y} = f'(x) \cdot \underbrace{dx}_{\text{differential of } x}}$$

dy → change in y

dx → change in x

Exp) Find differential of $y = x^3$ at $x = 2$.

$dy = ?$

$dy = f'(x) \cdot dx$ $\qquad$ $y = f(x) = x^3$

$dy = 3x^2 \cdot dx$ $\qquad$ $f'(x) = 3x^2$

$$dy\Big|_{x=2} = 3\cdot 2^2\cdot dx = 12\cdot dx$$

when x=2, a small change in x (dx) results in

a small change in y (dy)

Marginal Analysis in Economics

an area of economics concerned w/

how the change in the level of production

affects the cost, revenue and profit.

$x \to$ # of items produced

$C(x) \to$ cost function

(the cost of producing x units)

$mc(x) \to$ marginal cost function

(the additional cost of producing 1 more unit)

$\Delta C = mC(x) = C(x+1) - C(x)$ [Exact cost]

$mc(x) \approx C'(x) \cdot \underbrace{dx}$ [Estimate]

(cost of 1 more unit means $dx = 1$ unit)

$\underbrace{mc(x)} \approx C'(x)$

cost of producing the $(x+1)^{th}$ item

$R(x) \to$ revenue function

(the revenue obtained from producing x units)

$MR(x) \to$ marginal revenue function

(the additional revenue obtained from producing one more unit)

$R(x) = p(x) \cdot x$

$p(x) \to$ demand function (the market price)

$mR(x) = \Delta R = R(x+1) - R(x)$ (actual revenue)

$mR(x) \approx R'(x) \cdot \underbrace{\Delta x}$ (estimated revenue)

[dx = Δx] change in production (# of items)

Exp) $C(x) = \frac{1}{8}x^2 + 3x + 98$ (cost function)

$p(x) = \frac{1}{3}(75 - x)$ (price per item)

a) Find the marginal cost & revenue

a) $mc(x) \to$ marginal cost : $mc(x) \approx C'(x) \cdot dx$

$$mc(x) = C'(x) = \left(\frac{1}{8}x^2 + 3x + 98\right)'$$

$$= \frac{2}{8}x + 3 = \frac{x}{4} + 3$$

$MR(x) = R'(x) \cdot dx$

Recall: $R(x) = \underbrace{p(x)} \cdot x$

$R(x) = \underbrace{\left[\frac{1}{3} \cdot (75-x)\right]}_{p(x)} \cdot x = \frac{x}{3}(75-x)$

$= 25x - \frac{x^2}{3}$

$MR(x) = R'(x) = \left[25x - \frac{x^2}{3}\right]' = 25 - \frac{2}{3}x$

b) What's the exact cost of producing the 9th item?

The exact cost of the 9th item:

$\Delta C = \underbrace{C(9)}_{\text{Cost of 9 items}} - \underbrace{C(8)}_{\text{Cost of 8 items}}$

$C(x) = \frac{1}{8}x^2 + 3x + 98$

$\Delta C = \frac{1}{8} \cdot 9^2 + 3 \cdot 9 + 98 - \left(\frac{1}{8} \cdot 8^2 + 3 \cdot 8 + 98\right) = 5.125$

$\approx \$5.13$

c) Use marginal analysis to estimate the cost of producing the 9th item?

Use marginal analysis:

Est. the cost of the 9th item:

(change in cost as x increases from 8 to 9)

$C'(8) = \frac{8}{4} + 3 = 5$ (\$5)

Recall:

$C'(x) = \frac{x}{4} + 3$

d) What's the exact revenue obtained from selling the 9th unit?

$\Delta R = R(9) - R(8)$

$= 25 \cdot 9 - \frac{9^2}{3} - \left(25 \cdot 8 - \frac{8^2}{3}\right)$

$= 58/3 \approx \$19.33$

Recall:

$R(x) = p(x) \cdot x = 25x - \frac{x^2}{3}$

Use marginal analysis to estimate the revenue obtained from the 9th unit.

Recall: $R'(x) = -\frac{2}{3}x + 25$

$$R'(8) = \frac{-2}{3} \cdot 8 + \frac{25}{1}_{(3)} = \frac{-16}{3} + \frac{75}{3} = \frac{59}{3} \approx \$19.67$$

Actual revenue of \$19.33 vs.

estimated revenue of \$19.67

from the sale of the 9th unit.