4.1. Extreme Values

Note Title 3/23/2020

Goal: To investigate the behavior of different functions (such as finding the min/max value of functions or even if the min/max exists!). The process of finding the min/max of a function is called Optimization.

Definitions:

Absolute (Global) max: $f(c)$ is the absolute max. of function f on its domain D if $f(c) \geq f(x)$ for ALL x in its interval $[a, b]$.
($f(c)$ is the largest possible value of f)

Absolute (Global) Min: $f(c)$ is the absolute min. of function f on its domain D if $f(c) \le f(x)$ for ALL x in its interval $[a, b]$.
($f(c)$ is the smallest possible value of f)

Abs. min or max of f is called EXTREME VALUE or ABS. EXTREMUM.

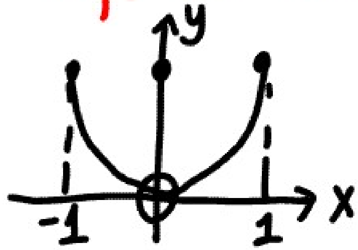
Plural of extremum is extrema or extremums.

Extreme Value Theorem (EVT)

If a function f is continuous on a closed, bounded interval I,

Then both abs. min and max. should exist!

Exp) A discontinuous function:



f is NOT continuous at $x=0$ in $[-1, 1]$.

Although it has an abs. max at $(-1,1)$ AND $(1,1)$, (Yes, it's OK. to have multiple abs. max/min points) it does not have an abs. min.

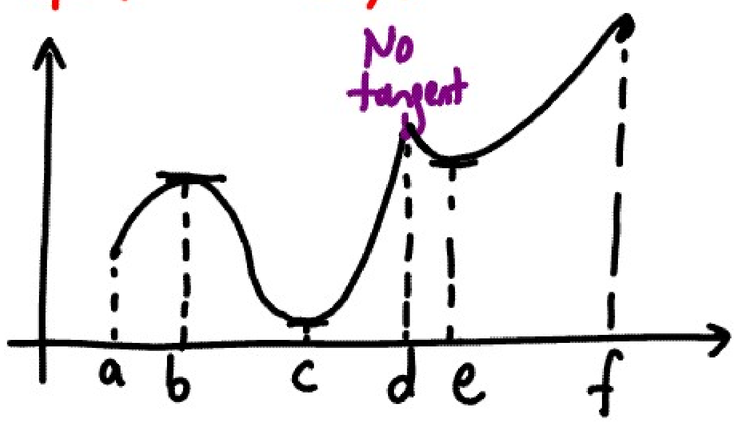
Relative (Local) max: If $f(c) \geq f(x)$ for all x in an open interval including c then f has a local max. at $x=c$.

Relative (Local) min: If $f(c) \leq f(x)$ for all x in an open interval including c then f has a local min. at $x=c$.

Together, they're called Relative Extrema.

What does that mean? An extremum at an endpoint (such as $x=a, b$ are endpoints in $[a,b]$) is NOT a relative extremum.

Example: Locate the extreme values of f on [a,f]



Graph of $f(x)$ on $[a,f]$

Highest point on f occurs at $x=f$.

Lowest point on f occurs at $x=c$.

Abs. max. value is $f(f)$

Abs. min. value is $f(c)$.

Peak (local max) — $f(b), f(d)$

Valley (local min) — $f(c), f(e)$

Note: $f(e) > f(b)$ which is ok!

Critical #/point: f is defined at $x=c$ and either $f'(c)=0$ or $f'(c)$ DNE; then $x=c$ is a critical #, and $(c, f(c))$ is a critical point.

If $f(c)$ is a local extremum, Then c is a critical # of f.

BUT; it doesn't imply that a local extremum must occur at each critical #.

E.g: $f(x)=x^3$, $f'(x)=3x^2=0 \Rightarrow f'(0)=0$

So; 0 is a critical #. But; no local extremum at $x=0$!

Procedure for finding Abs. Extrema

$f(x)$ is a continuous function on $[a,b]$

① find critical #s of f: [$f'(x)=0$ or DNE]

identify the endpoints [$x=a, b$]

② evaluate f at endpoints a and b

and at each critical # c.

compare the values in step 2.

smallest value is abs. min.

largest value is abs. max.

Expl) Find the abs. min/max. of

$f(x)=2x^3-6x^2+1$ on $[1,4]$

Step1) All polynomials are cont. in their domain.

critical #s: $f'(x)=0$ or DNE

$f'(x)=6x^2-12x=6x(x-2)=0 \Rightarrow \cancel{x=0}, x=2$

not in $[1,4]$

$\underbrace{\text{endpoints } x=1,4}$ on $[1,4]$

Step2)

	x	$f(x)=2x^3-6x^2+1$	
critical #	2	$2\cdot2^3-6\cdot2^2+1=-7$	$\rightarrow$ min
endpoints	1	$2\cdot1^3-6\cdot1^2+1=-3$	
	4	$2\cdot4^3-6\cdot4^2+1=33$	$\rightarrow$ max

Abs. min value is -7 at $x=2$.

Abs. max. value is 33 at $x=4$.

Exp2) You try it!

Find abs. extreme values of $f(x) = x^3 + 3x^2 - 24x - 4$ on $[-4, 4]$

Poll answers:

A) abs. max. value is -32

B) abs. max. value is 76

C) abs. max. value is 12

D) None of the above

Exp2) Solution

Find abs. extreme values of $f(x)=x^3+3x^2-24x-4$ on $[-4,4]$

① $f(x)$ is cont. on $[-4,4]$

$f'(x)=0$ or DNE to find critical #s

$f'(x)=3x^2+6x-24=3(x^2+2x-8)$

4 −2

$f'(x)=3(x+4)(x-2)=0 \Rightarrow \underbrace{x=-4,\ x=2}_{\text{critical \#s on } [-4,4]}$

endpoints: $x=-4, 4$

②

x	$f(x)=x^3+3x^2-24x-4$	
-4	$-64+3\cdot16-24\cdot-4-4=76$	MAX
2	$2^3+3\cdot2^2-24\cdot2-4=-32$	MIN
4	$4^3+3\cdot4^2-24\cdot4-4=12$	

Abs. max. value is 76. (B)

Exp) Find the abs. extrema of
$f(x) = x^{2/3}(5-2x)$ on $[-1,2]$

Solution:

1 $f(x) = 5x^{2/3} - 2 \cdot x^{5/3}$

$$f'(x) = 5 \cdot \frac{2}{3} \cdot x^{-1/3} - 2 \cdot \frac{5}{3} \cdot x^{2/3}$$

$$= \frac{10}{3} \cdot x^{-1/3} - \frac{10}{3} \cdot x^{2/3}$$

$$= \frac{10}{3} x^{-1/3} \left(1 - x^{3/3}\right) = \frac{10(1-x)}{3\sqrt[3]{x}}$$

critical #s: $f'(x) = 0$ or DNE

$$\left.\begin{array}{l} f'(x) = 0 \Rightarrow 1-x=0 \Rightarrow x=1 \\ f'(x) \text{ DNE} \Rightarrow \sqrt[3]{x} = 0 \Rightarrow x=0 \end{array}\right\} \text{critical #s on } [-1,2]$$

endpoints $x=-1$, $x=2$

(2)

x	$f(x)=x^{2/3}(5-2x)$	
0	0	MIN
1	$1^{2/3}(5-2)=3$	
-1	$(-1)^{2/3}(5+2)=7$	MAX
2	$2^{2/3}(5-4)=2^{2/3}$	

The abs. max. value is 7 at $x=-1$.

The abs. min. value is 0 at $x=0$.

Exp) Find the abs. extreme values of

$$f(x) = x - \frac{2x}{x+2} \text{ on } [-1,4]$$

① $f(x)$ is cont. on $[-1,4]$

Although $x=-2$ is a point of discont.

it's NOT on the given interval.

$$f'(x) = 1 - \frac{4}{(x+2)^2}$$

critical #s: $f'(x) = 1 - \frac{4}{(x+2)^2} = 0$ or DNE

$$f'(x)=0 \Rightarrow 1 = \frac{4}{(x+2)^2} \Rightarrow 4 = (x+2)^2$$

$x+2=2$ $\quad$ $x+2=-2$

$\underbrace{x=0}_{\text{only critical \#}}$ $\quad$ $\cancel{x=-4}$ not in $[-1,4]$

$f'(x)$ DNE: recall $x=-2$ is NOT in $[-1,4]$

endpoints are $x=-1$, $x=4$

(2)

x	$f(x) = x - \frac{2x}{x+2}$
-1	$-1 - \frac{2(-1)}{-1+2} = 1$
4	$4 - \frac{2 \cdot 4}{4+2} = \left(4 - \frac{8}{6}\right)$ max
0	0 min

Abs. min value is 0 at $x=0$

Abs. max. value is $\left(4 - \frac{8}{6} = \frac{8}{3}\right)$ at $x=4$

You try it!

Find abs. min/max values for

$f(x) = \frac{9}{x} + x - 3$ on $[1, 9]$

Poll choices:

A) Abs. min at $x = 3$

B) Abs. min. at $x = 1$

C) Abs. min at $x = 9$

D) Abs. min at $x = 0$

E) Abs. min at $x = -3$

You try it! Solution

Find abs. min/max values for

$f(x) = \frac{9}{x} + x - 3$ on $[1, 9]$

Step 1 $f(x)$ is NOT continuous at $x=0$
however, 0 is not in $[1, 9]$

$f(x) = 9 \cdot x^{-1} + x - 3$

$f'(x) = -9x^{-2} + 1$

critical #s $\rightarrow$ $f'(x) = 0$ or DNE

$f'(x) = 0 \Rightarrow \frac{-9}{x^2} + 1 = 0 \Rightarrow \frac{-9}{x^2} = -1 \Rightarrow 9 = x^2$

$x = 3$ $\cancel{x = -3}$

not in $[1,9]$

only $x=3$ is a critical #

$f'(x)$ DNE : $x=0$, however, as stated before $x=0$ is NOT in $[1,9]$

endpoints: $x=1, 9$

Step2

x	$f(x)=\frac{9}{x}+x-3$	
critical # 3	$f(3)=\frac{9}{3}+3-3=3$	MIN
endpoints 1	$f(1)=9+1-3=7$	MAX
9	$f(9)=\frac{9}{9}+9-3=7$	MAX

The absolute max. is 7, the abs. max points are: $(1,7)$ and $(9,7)$.

(It's OK that the abs. max. values occur at multiple x-values)

The abs. min. is 3, the abs. min point is: $(3,3)$