Note Title 3/9/2020

4.4. Recap

Limits at $\pm\infty$

$$\lim_{x \to \pm\infty} f(x) = \text{finite \#} \text{ or } \pm\infty$$

Procedure

$\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} \rightarrow$ ① identify the highest exponent of x in the denominator

② factor it out both from numerator & denominator

③ use: $\lim_{x \to \pm\infty} \left(\frac{A}{x^r}\right) = 0$ to eval.

Infinite Limits

$$\lim_{x \to C} f(x) = \mp\infty$$

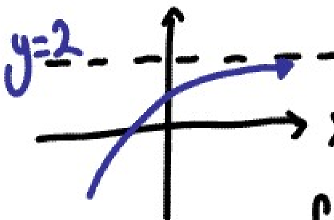
Procedure: when "DSP" $\frac{\text{"non-zero \#"}}{0}$ is infinite

do sign analysis for numerator & denominator to determine the sign of infinity as $\pm\infty$.

Horizontal Asymptote

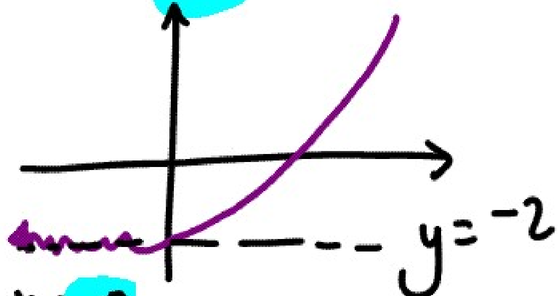
The line $\boxed{y=L}$ is a Horizontal Asymptote

if $\lim_{x\to\infty} f(x)=L$ or $\lim_{x\to-\infty} f(x)=L$.



$x\to+\infty$

$f(x)\to 2$



$x\to-\infty$

$f(x)\to -2$ graph may cross H.A.

* A function may have at most 2 H.A

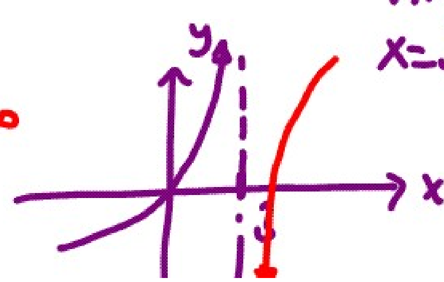
Vertical Asymptote

The line $\boxed{x=c}$ is a Vertical Asymptote if:

$\lim_{x\to c^-} f(x$ or $\lim_{x\to c^+} f(x)$ is infinite

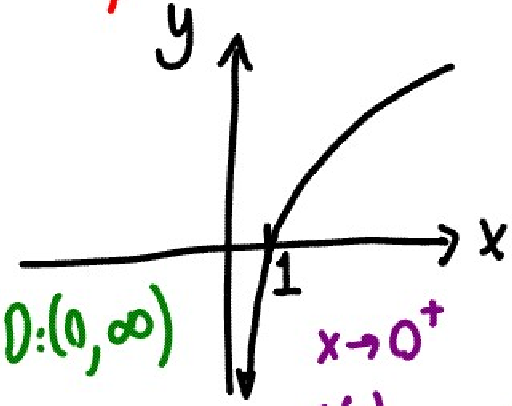
V.A.
x=3

$\lim_{x\to 3^-} f(x)=+\infty$ $\lim_{x\to 3^+} f(x)=-\infty$



* graph can NOT cross V.A.

Exp) Find all vertical asymptotes of $f(x)=\ln x$



$D:(0,\infty)$

$x\to 0^+$, $f(x)\to -\infty$ $\Big\}$ $\lim\limits_{x\to 0^+} f(x) = -\infty$

Exp) Evaluate $\lim\limits_{x\to 3} \dfrac{4x}{x-3}$

$$\lim_{x\to 3^-}\left(\frac{4x}{x-3}\right) \overset{\text{"DSP"}}{=} \frac{4\cdot 3}{3^- - 3} = \frac{12}{0^-} = -\infty$$

$\hookrightarrow 2.999\ldots$

$$\lim_{x\to 3^+}\left(\frac{4x}{x-3}\right) \overset{\text{"DSP"}}{=} \frac{4\cdot 3}{3^+ - 3} = \frac{12}{0^+} = +\infty$$

$\hookrightarrow 3.000\ldots 1$

Exp) Find ALL vertical asymptotes:
(check one-sided limits at each V.A.)

first simplify by factoring

then identify V.A

$$f(x) = \frac{x^2-4x+3}{x^2-6x+9}$$

$$= \frac{\cancel{(x-3)}(x-1)}{(x-3)^{\cancel{2}}} = \frac{x-1}{x-3} \rightarrow x-3=0$$

$x=3$ V.A.

$$\lim_{x\to 3^-}\left(\frac{x-1}{x-3}\right) \overset{\text{"DSP"}}{=} \frac{3-1}{3^- -3} = \frac{2}{0^-} = -\infty$$

$$\lim_{x\to 3^+}\left(\frac{x-1}{x-3}\right) \overset{\text{"DSP"}}{=} \frac{3-1}{3^+ -3} = \frac{2}{0^+} = +\infty$$

4.5. L'Hôpital's Rule (LR)

Procedure: IF $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} \overset{\text{"DSP"}}{=} \underbrace{\text{"}\dfrac{0}{0}\text{" or } \pm\text{"}\dfrac{\infty}{\infty}\text{"}}_{\text{(primary) indeterminate forms}}$

THEN $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} = L$

where L is either a finite # or $\pm\infty$.

Indeterminate forms: are expressions that their value cannot be determined w/out further analysis.

$\overset{H}{=}$ means L.R. is used

Important Notes

* NOT a quotient rule (differentiate $f(x)$, $g(x)$ separately)

* LR can be applied repeatedly
* justify the use of L.R. then use it!
* LR applies to one-sided limits and limits at ∞ as well
  $(x \to c^-, x \to c^+, x \to \mp\infty)$

Secondary Indeterminate forms:

$$\underbrace{1^\infty, 0^0, \infty^0,}_{} \quad \infty - \infty, \; 0 \cdot \infty$$

procedure w/
using the natural log properties

Exp) Evaluate $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)$

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right) \overset{\text{"DSP"}}{=} \frac{\sin 0}{0} = \text{"}\frac{0}{0}\text{"} \checkmark$$

$$\overset{H}{=} \lim_{x\to 0} \frac{(\sin x)'}{x'} = \lim_{x\to 0} \frac{\cos x}{1} \overset{\text{"DSP"}}{=} \cos(0) = 1$$

Exp) Evaluate $\lim_{x\to 2} \frac{x^7-128}{x^3-8}$

$$\lim_{x\to 2}\left(\frac{x^7-128}{x^3-8}\right) \overset{\text{"DSP"}}{=} \frac{2^7-128}{2^3-8} = \text{"}\frac{0}{0}\text{"} \checkmark$$

$$\overset{\boxed{H}}{=} \lim_{x\to 2} \frac{(x^7-128)'}{(x^3-8)'} = \lim_{x\to 2} \frac{7x^6}{3x^2} = \lim_{x\to 2}\left(\frac{7}{3}x^4\right)$$

use L.R.

$$\overset{\text{"DSP"}}{=} \frac{7}{3}\cdot 2^4 = \frac{7}{3}\cdot 16 = \frac{112}{3}$$

Exp) Evaluate $\lim_{x \to 0} \left(\frac{1-\cos x}{\sec x} \right)$

distractor

a) ~~1~~ $\boxed{\text{b) } 0}$ c) ∞ d) $-\infty$ e) DNE

$$\lim_{x \to 0} \left(\frac{1-\cos x}{\sec x} \right) \overset{\text{"DSP"}}{=} \frac{1-\cos 0}{\sec 0} = \frac{1-1}{1} = \frac{0}{1} = 0$$

Recall: $\sec x = \frac{1}{\cos x}$

No need for L.R.!

Exp) Evaluate $\lim_{x \to 0} \left(\frac{x-\sin x}{x^3} \right)$. $\overset{\text{"DSP"}}{=} \frac{0-\sin 0}{0^3} = \text{"}\frac{0}{0}\text{"}$

$$\overset{H}{=} \lim_{x \to 0} \frac{(x-\sin x)'}{(x^3)'} = \lim_{x \to 0} \frac{(1-\cos x)}{3x^2} \overset{\text{"DSP"}}{=} \frac{1-\cos 0}{3 \cdot 0^2} = \text{"}\frac{0}{0}\text{"} ✓$$

$$\overset{H}{=} \lim_{x \to 0} \frac{(1-\cos x)'}{(3x^2)'} \boxed{= \lim_{x \to 0} \frac{\sin x}{6x}} \overset{\text{"DSP"}}{=} \frac{\sin 0}{6 \cdot 0} = \text{"}\frac{0}{0}\text{"}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{(\sin x)'}{(6x)'} = \lim_{x\to 0} \frac{\cos x}{6} \overset{\text{"DSP"}}{=} \frac{\cos 0}{6} = \boxed{\frac{1}{6}}$$

Exp) Evaluate $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$.

$$\overset{\text{"DSP"}}{=} \frac{\sqrt{\infty^2+1}}{\infty} = \text{"}\frac{\infty}{\infty}\text{"} \checkmark$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{\cancel{2}}(x^2+1)^{-1/2}\cdot \cancel{2}x}{1} = \lim_{x\to\infty} \left[(x^2+1)^{-1/2}\cdot x\right]$$

$$= \lim_{x\to\infty} \left[\frac{x}{(x^2+1)^{1/2}}\right] \overset{\text{"DSP"}}{=} \text{"}\frac{\infty}{(\infty^2+1)^{1/2}}\text{"} = \text{"}\frac{\infty}{\infty}\text{"}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{\cancel{2}}\underbrace{(x^2+1)^{-1/2}}\cdot \cancel{2}x} = \lim_{x\to\infty} \frac{(x^2+1)^{1/2}}{x}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$$ (this was our original pb.)

L.R. didn't work!

We have to use the procedure from 4.4 (Limits at Infinity)

$$\lim_{x\to\infty} \left(\frac{\sqrt{x^2+1}}{x}\right) = \lim_{x\to\infty} \frac{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x\,(1)}$$

(since $x \to +\infty$ $\sqrt{x^2} = |x| = x$)

$$\lim_{x\to\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}}{x} = \lim_{x\to\infty} \frac{\cancel{x}\left(\sqrt{1+\frac{1}{x^2}}\right)}{\cancel{x}}$$

$$\lim_{x\to\infty} \left(\sqrt{1+\frac{1}{x^2}}\right) \overset{\text{"DSP"}}{=} \text{"}\sqrt{1+\cancel{\frac{1}{\infty^2}}^{\,0}}\text{"} = 1$$

(Other / Secondary) Indeterminate Forms

$$\underbrace{1^\infty,\ 0^0,\ \infty^0}_{\text{use ln properties}} \qquad \infty-\infty \qquad 0\cdot\infty$$

Procedure:

1) "DSP" to obtain an indeterminate form
2) Re-write "other indeterminate forms" into "primary indeterminate form"
3) Use L.R.

Exp) Evaluate $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ $\boxed{1^\infty}$

DSP $\quad \overset{\text{"DSP"}}{=} \left(1+\cancel{\frac{1}{\infty}}^{\,0}\right)^\infty = 1^\infty$ ✓

Set the prb. to y $\quad y = \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

Take the "ln" of each side

$\boxed{y=?}$

↓

Eval. y NOT lny!

$$\ln y = \ln \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x}$$

(Swap ln & lim / Limit of a log property)

$$\ln y = \lim_{x\to\infty} \left(\ln \left(1+\frac{1}{x}\right)^{x}\right)$$

(property of logarithms)

$$= \lim_{x\to\infty} \left(x \cdot \ln\left(1+\frac{1}{x}\right)\right)$$

$$\left(x = \frac{1}{\frac{1}{x}}\right) \longleftarrow \text{Algebra rule: } x = 1/(1/x)$$

$$= \lim_{x\to\infty} \left(\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}\right)$$

$$\overset{\text{"DSP"}}{=} \frac{\ln\left(1+\cancel{\frac{1}{\infty}}^{\,0}\right)}{\cancel{1/\infty}^{\,0}} = \text{"}\frac{0}{0}\text{"} \checkmark$$

$$\overset{H}{=} \lim_{x\to\infty} \left(\frac{\left[\ln\left(1+\frac{1}{x}\right)\right]'}{\left[x^{-1}\right]'}\right)$$

Recall: $[\ln(u)]' = \frac{u'}{u}$

$$= \lim_{x\to\infty} \left(\frac{\frac{\left(1+\frac{1}{x}\right)'}{\left(1+\frac{1}{x}\right)}}{-1\cdot x^{-2}} \right) = \lim_{x\to\infty} \frac{\frac{\left(\cancel{-x^{-2}}\right)^{1}}{\left(1+\frac{1}{x}\right)}}{\cancel{-x^{-2}}_{1}}$$

$$= \lim_{x\to\infty} \left(\frac{1}{\left(1+\frac{1}{x}\right)} \right) \overset{\text{"DSP"}}{=} \text{"} \frac{1}{1+\cancel{\frac{1}{\infty}}^{0}} \text{"} = 1$$

Be careful!

$$\ln y = 1 \Rightarrow y = e^{1}$$

Final answer!

Exp) Evaluate $\lim_{x\to 0^+} \left(x^x\right)$

$$\overset{\text{"DSP"}}{=} \boxed{0^0} \checkmark$$

$(y = ?)$ $\quad \ln y = \ln \lim_{x\to 0^+} (x^x)$

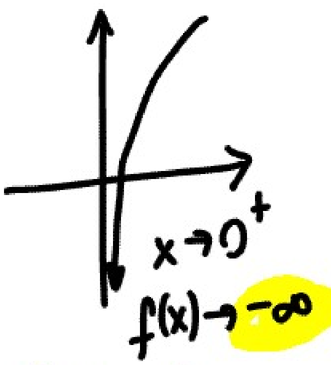
$$\ln y = \lim_{x\to 0^+} \left(\ln (x^x) \right)$$

$$\ln y = \lim_{x\to 0^+} \left(x \cdot \ln(x) \right) \qquad \text{"}\frac{0}{0}\text{"}, \text{"}\frac{\infty}{\infty}\text{"}$$

$$= \lim_{x\to 0^+} \left(\frac{\ln(x)}{\frac{1}{x}} \right) \overset{\text{"DSP"}}{=} \frac{-\infty}{\frac{1}{0^+}}$$

$$= \text{"}\frac{-\infty}{\infty}\text{"} \checkmark$$

Recall:



Graph of $f(x) = \ln x$

$$\overset{H}{=} \lim_{x\to 0^+} \left(\frac{1/x}{-1/x^2} \right)$$

"keep change flip"

$$\overset{H}{=} \lim_{x\to 0^+} \left(\frac{1}{\cancel{x}} \cdot \frac{-x^{\cancel{2}}}{1} \right)$$

$$= \lim_{x \to 0^+} (-x) \overset{\text{"DSP"}}{=} 0$$

Be careful!

$$\ln y = 0 \Rightarrow \underbrace{y = e^0 = 1}_{\text{Final answer}}$$

Exp) Evaluate $\lim_{x \to \infty} (\ln x)^{1/x}$

"DSP" $\lim_{x \to \infty} (\ln x)^{1/x} \overset{\text{"DSP"}}{=} \text{"}(\ln(\infty))^{1/\infty}\text{"}$

$$= \boxed{\infty^0}$$

Set it to y & take "ln"

$$y = \lim_{x \to \infty} (\ln x)^{1/x}$$

Swap ln & lim

$$\ln y = \ln\left(\lim_{x\to\infty} (\ln x)^{1/x}\right)$$

Use properties of logarithms

$$\ln y = \lim_{x\to\infty}\left(\ln (\ln x)^{1/x}\right)$$

$$\ln y = \lim_{x\to\infty}\left(\frac{1}{x}\cdot \ln(\ln x)\right)$$

"DSP again"

$$\lim_{x\to\infty}\left(\frac{\ln(\ln x)}{x}\right) \overset{\text{"DSP"}}{=} \text{"}\frac{\ln(\ln\infty)}{\infty}\text{"} = \text{"}\frac{\infty}{\infty}\text{"}$$

$$\ln y \overset{H}{=} \lim_{x\to\infty}\frac{[\ln(\ln x)]'}{x'}$$

$$= \lim_{x\to\infty}\left(\frac{\frac{\frac{1}{x}}{\ln x}}{1}\right) = \lim_{x\to\infty}\left(\frac{1}{x\cdot\ln x}\right)$$

$$\overset{\text{"DSP"}}{=} \text{"}\frac{1}{\infty \cdot \ln \infty}\text{"} = \text{"}\frac{1}{\infty}\text{"} = 0$$

Be careful!

$$\ln y = 0 \Rightarrow y = e^0 = 1$$ final answer

Exp) Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$

"DSP" $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right) \overset{\text{"DSP"}}{=} \boxed{(\infty - \infty)}$

(sinx) (x)

"Re-write to obtain $\frac{0}{0}$ or $\frac{\infty}{\infty}$"

$$\lim_{x \to 0^+} \left(\frac{\sin x - x}{x \cdot \sin x} \right) \overset{\text{"DSP"}}{=} \frac{\sin 0 - 0}{0 \cdot \sin 0} = \text{"}\frac{0}{0}\text{"}$$

"use L.R"

$$\lim_{x \to 0^+} \left(\frac{\cos x - 1}{1 \cdot \sin x + x \cdot \cos x} \right) \overset{\text{"DSP"}}{=} \text{"}\frac{0}{0}\text{"}$$

"use L.R. again"

$$\lim_{x \to 0^+} \left(\frac{-\sin x}{\cos x + 1 \cdot \cos x + x \cdot (-\sin x)} \right)$$

$$\overset{\text{"DSP"}}{=} \frac{-\sin 0}{\cos 0 + \cos 0 + 0 \cdot (-\sin 0)}$$

$$= \frac{0}{1 + 1 + 0} = \frac{0}{2} = 0$$

Exp) Evaluate $\lim_{x \to \left(\frac{\pi}{2}\right)^-} \left[\left(x - \frac{\pi}{2} \right) \cdot \tan x \right]$

"DSP" $\lim_{x \to \left(\frac{\pi}{2}\right)^-} \left[\left(x - \frac{\pi}{2} \right) \cdot \tan x \right] \overset{\text{"DSP"}}{=} \boxed{\text{"}0 \cdot \infty\text{"}}$

Recall: "$\tan\left(\frac{\pi}{2}\right)^- = \frac{\sin\left(\frac{\pi}{2}\right)^-}{\cos\left(\frac{\pi}{2}\right)^-} = \frac{1}{0} = \infty$"

"Re-write to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$"

$$\lim_{x \to \left(\frac{\pi}{2}\right)^-} \left[\left(x - \frac{\pi}{2}\right) \cdot \frac{1}{\cot x} \right]$$

$$= \lim_{x \to \left(\frac{\pi}{2}\right)^-} \left(\frac{\left(x - \frac{\pi}{2}\right)}{\cot x} \right) \overset{\text{"DSP"}}{=} \text{"}\frac{0}{0}\text{"}$$

"Use L.R"

$$\overset{H}{=} \lim_{x \to \left(\frac{\pi}{2}\right)^-} \left(\frac{1}{-\csc^2 x} \right) = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \left(-\sin^2 x\right)$$

Recall: $\csc x = \frac{1}{\sin x}$

$$\overset{\text{"DSP"}}{=} -\sin^2\left(\frac{\pi}{2}\right) = -1$$