3/26 midterm #2 Review

Note Title 3/25/2020

Sections 3.6, 3.7, 3.8, 4.4, 4.5

3.6 Find an equation of the normal line to the curve $x^2 \cdot \sqrt{y-2} = y^2 - 3x - 5$ at $(1,3)$

Solution:

Differentiate w/ respect to x:

$$2x \cdot \sqrt{y-2} + x^2 \cdot \frac{1}{2}(y-2)^{-1/2} \cdot \frac{dy}{dx} = 2y \cdot \frac{dy}{dx} - 3$$

No need to get dy/dx by itself first; subs. $(1,3)$

$$2 \cdot 1 \sqrt{3-2} + 1^2 \cdot \frac{1}{2}(3-2)^{-1/2} \cdot \frac{dy}{dx} = 2 \cdot 3 \cdot \frac{dy}{dx} - 3$$

$$2 + \frac{1}{2} \cdot \frac{dy}{dx} = 6 \cdot \frac{dy}{dx} - 3$$

$$6\frac{dy}{dx} - \frac{1}{2}\frac{dy}{dx} = 2+3 \Rightarrow \frac{2}{11}\cdot\frac{11}{2}\cdot\frac{dy}{dx} = 5\cdot\frac{2}{11}$$

$$\left.\frac{dy}{dx} = \frac{10}{11}\right\} m_{tan}$$

$$\boxed{m_{normal} = \frac{-1}{m_{tan}}} = -\frac{11}{10}$$

Eq. of the normal line to curve at (1,3):

$$y - 3 = \frac{-11}{10}(x-1)$$

You try it!

Find the eq. of the tangent line to graph of

$x^2+(y-x)^3=9$ at $x=1$.

A) $y+3=\frac{-6}{5}(x-1)$

B) $y-3=\frac{5}{6}(x-1)$

C) $y-3=\frac{2}{3}(x-1)$

D) None of the above

Solution: Differentiate both sides wrt x:

$$2x + 3(y-x)^2 \cdot \left(\frac{dy}{dx} - 1\right) = 0$$

Subs. $(x, y) \to (1, 3)$

$x = 1$ subs. in
$x^2 + (y-x)^3 = 9$
yields $y = 3$

$$2 \cdot 1 + 3(3-1)^2 \left(\frac{dy}{dx} - 1\right) = 0$$

$$2 + 3 \cdot 4 \left(\frac{dy}{dx} - 1\right) = 0$$

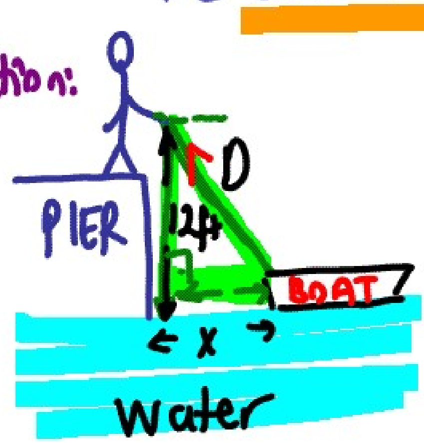
$$12\left(\frac{dy}{dx} - 1\right) = -2$$

$$\frac{dy}{dx} - 1 = -\frac{1}{6} \Rightarrow \frac{dy}{dx} = \frac{5}{6} = m_{tan}$$

Eq. of the tangent line: $y - 3 = \frac{5}{6}(x-1)$

3.7 Exp) A person standing at the end of a pier 12 ft. above the water and is pulling in a rope attached to a rowboat at the rate of 6 ft./min. How fast is the boat moving when it's 16 ft. from pier?

Solution:



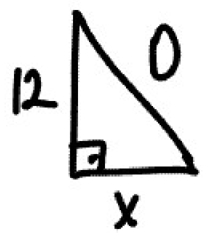
when $\frac{dD}{dt} = -6 \frac{ft}{min}$

(D is decreasing by time)

AND $x = 16$ ft.

what's $\frac{dx}{dt} = ?$

Use the right Δ:



$\Rightarrow 12^2 + x^2 = D^2$

$$144 + x^2 = D^2$$

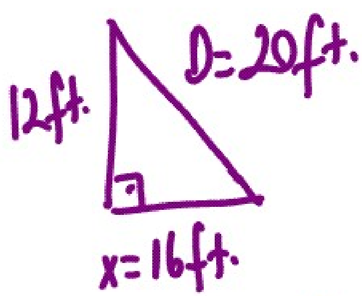
Differentiate both sides wrt time

use units as a hint!

$$0 + 2x \cdot \frac{dx}{dt} = 2D \cdot \frac{dD}{dt}$$

Subs. given info @ specific time:

$$x = 16\text{ft.}, \quad \frac{dD}{dt} = -6\,\frac{\text{ft}}{\text{min}}$$



(3-4-5 special tri.)

$$x \cdot \frac{dx}{dt} = D \cdot \frac{dD}{dt}$$

$$16 \cdot \frac{dx}{dt} = 20 \cdot (-6)$$

$$\frac{dx}{dt} = \frac{-120}{16} = -7.5\,\frac{\text{ft}}{\text{min.}}$$

Distance between the boat & pier is decreasing.

3.8 Use linear approximation to estimate:

$$(1.005)^{50}$$

Solution: Let $f(x) = x^{50}$. The goal is to estimate $f(1.005)$ by using the tangent line to $f(x)$ at $\underline{a=1}$

$a \to$ known, easy to compute value, close to unknown value

$$L(x) = f(a) + f'(a)(x-a)$$

$\underline{a=1}$; $f(a) = f(1) = 1^{50} = 1$, $f'(x) = 50 \cdot x^{49}$

$f'(1) = 50 \cdot 1^{49} = 50$

Subs. the in the $L(x)$ eq:

$$L(x) = 1 + 50(x-1)$$

$$L(1.005) = 1 + 50(1.005 - 1)$$

$$= 1 + 50(0.005) = 1 + 0.25 = 1.25$$

With calculator: $(1.005)^{50} \approx 1.2832$

3.8 A manufacturer's total cost is

$$C(x) = 200 + 0.5x + \frac{10,000}{x}$$

dollars, where x is the # of units produced.

a) use marginal analysis to estimate the cost of manufacturing the 201st item

Solution: The cost is estimated by:

$$\Delta C \approx C'(200) \cdot 1$$

↳ diff. between 200th vs. 201th item

$$C'(x) = 0.5 - 10,000x^{-2}$$

$$C'(200) = 0.5 - 10,000 \cdot (200)^{-2} = 0.25 \ \$$$

b) Compute the exact cost of manufacturing the 201th item?

The exact cost is computed by:

$$\Delta C = C(201) - C(200)$$

$$= 200 + 0.5 \cdot 201 + \frac{10{,}000}{201} - \left(200 + 0.5 \cdot 200 + \frac{10{,}000}{200}\right)$$

(w/ calculator) $\simeq 0.251243$ (round to nearest cents)

$\simeq 0.25$

(It's a coincidence that the computed vs. estimated costs are equal..)

4.4 Find ALL vertical asymptotes of $f(x) = \dfrac{2x^2-5x+7}{x^2-9}$. Justify your answer.

Solution: set the denominator equal to 0.

$$x^2-9=0 \Rightarrow x = \mp 3$$

Are $x=\mp 3$ really the V.A? Justify.

check limits from both sides.

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{2x^2-5x+7}{x^2-9} \overset{\text{"DSP"}}{=} \underbrace{\text{"}\frac{10}{0}\text{"}}$$

"$\dfrac{\text{non-zero \#}}{0}$" means V.A

$x\to 3, f(x)\to \mp\infty$

Next, analyze the sign of ∞ as $\pm\infty$?

Do a sign analysis on each factor of f(x):

$$\lim_{x\to 3^-}\left(\frac{2x^2-5x+7}{(x-3)(x+3)}\right) \overset{\text{"DSP"}}{=} \text{"}\frac{10}{\underbrace{0^-}_{\text{neg. \# very close to 0}}\cdot 6}\text{"} = -\infty$$

Note both 10, 6 are pos.

$$\lim_{x\to 3^+}\left(\frac{2x^2-5x+7}{(x-3)(x+3)}\right) \overset{\text{"DSP"}}{=} \text{"}\frac{10}{\underbrace{0^+}_{\text{pos. \# very close to 0}}\cdot 6}\text{"} = +\infty$$

Now, lets check $x\to -3$

$$\lim_{x\to -3^-}\left(\frac{2x^2-5x+7}{(x-3)(x+3)}\right) \overset{\text{"DSP"}}{=} \text{"}\frac{40}{-6\cdot \underbrace{0^-}_{\text{neg. \# very close to 0}}}\text{"} = +\infty$$

$$\lim_{x\to -3^+}\left(\frac{2x^2-5x+7}{(x-3)(x+3)}\right) \overset{\text{"DSP"}}{=} \text{"}\frac{40}{-6\cdot \underbrace{0^+}_{\text{pos. \# very close to 0}}}\text{"} = -\infty$$

Exp) Find ALL horizontal asymptotes of:

$$f(x) = \frac{\sqrt{169x^2+x}}{x}$$

Solution: To calculate the limits at $\pm\infty$, identify the highest exponent of x in both the numerator & denominator, then factor out from both:

$$\lim_{x\to\infty}\left(\frac{\sqrt{169x^2+x}}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2\left(169+\frac{1}{x}\right)}\right)}{x \cdot 1}$$

$$= \lim_{x\to\infty}\left(\frac{|x|\sqrt{169+\frac{1}{x}}}{x}\right)$$

Recall:

$$\underbrace{\sqrt{x^2} = |x|}$$

$\sqrt{x^2} = x$ for $x \geq 0$

$\sqrt{x^2} = -x$ for $x < 0$

$$= \lim_{x \to \infty} \left(\frac{x \cdot \sqrt{169 + \frac{1}{x}}}{x} \right) = \lim_{x \to \infty} \left(\sqrt{169 + \frac{1}{x}} \right)$$

$$\overset{\text{"DSP"}}{=} \sqrt{169 + 0}$$

$$= 13$$

Recall:

$$\lim_{x \to \pm\infty} \left(\frac{A}{x^r} \right) = 0$$

$A \to$ any real #
$r \to$ pos. rational #

$$\lim_{x \to -\infty} \left(\frac{|x| \sqrt{169 + \frac{1}{x}}}{x} \right)$$

$|x| = -x$ since $x < 0$
$(x \to -\infty)$

$$\lim_{x \to -\infty} \left(\frac{-x \cdot \sqrt{169 + \frac{1}{x}}}{x} \right) = \lim_{x \to -\infty} \left(- \sqrt{169 + \frac{1}{x}} \right) = -13$$

H.A. $y = 13, y = -13$

You try it!

Find all horizontal asymptotes of

$$f(x) = \frac{x^3+8}{4x^3+\sqrt{64x^6+4}}$$

A) $y = \pm\infty$

B) $y = -1/4$

C) $y = 1/4$

D) $y = -1/4,\ 1/12$

E) $y = 1/4,\ -1/12$

Solution: $\lim_{x\to\infty}\left(\dfrac{x^3\left(1+\frac{8}{x^3}\right)}{x^3\left(4+\frac{\sqrt{64x^6+4}}{x^3}\right)}\right)$

$$\lim_{x\to\infty}\left(\frac{\left(1+\frac{8}{x^3}\right)}{4+\frac{\sqrt{x^6\left(64+\frac{4}{x^6}\right)}}{x^3}}\right)$$

$\to$ as $x\to\infty$

$\sqrt{x^6}=|x^3|=x^3$

$$\lim_{x\to\infty}\left(\frac{\left(1+\frac{8}{x^3}\right)}{4+\frac{x^3\sqrt{64+\frac{4}{x^6}}}{x^3}}\right)=\lim_{x\to\infty}\left(\frac{\left(1+\frac{8}{x^3}\right)}{4+\sqrt{64+\frac{4}{x^6}}}\right)$$

$$\overset{\text{"DSP"}}{=}\frac{1+0}{4+\sqrt{64+0}}=\frac{1}{4+8}=\frac{1}{12}\qquad y=\frac{1}{12}$$

$$\lim_{x\to-\infty}\left(\frac{\left(1+\frac{8}{x^3}\right)}{4+\frac{\sqrt{x^6\left(64+\frac{4}{x^6}\right)}}{x^3}}\right)$$

$\rightarrow$ as $x\to-\infty$

$x<0$

$\sqrt{x^6}=|x^3|=-x^3$

$$\lim_{x\to-\infty}\left(\frac{\left(1+\frac{8}{x^3}\right)}{4+\frac{(-x^3)\sqrt{64+\frac{4}{x^6}}}{x^3}}\right)$$

$$\lim_{x\to-\infty}\left(\frac{\left(1+\frac{8}{x^3}\right)}{4-\sqrt{64+\frac{4}{x^6}}}\right)\overset{\text{"DSP"}}{=}\frac{1}{4-8}=-\frac{1}{4}$$

H.A. are $y=\frac{-1}{4}$, $y=\frac{1}{12}$

Justify L.R: $\frac{\infty}{\infty}, \frac{0}{0} \Leftarrow 1^{\infty}, 0^{0}, \infty^{0}, \infty-\infty, 0\cdot\infty$

4.5 Exp) Evaluate the limit or show it doesn't exist. Show all work.

a) $\lim\limits_{x\to 0} \frac{\sin^2 x}{\sin(x^2)} \overset{\text{"DSP"}}{=} \text{"}\frac{0}{0}\text{"}$ L.R. ✓

$\overset{H}{=} \lim\limits_{x\to 0}\left(\frac{\cancel{2}\cdot\sin x\cdot\cos x}{\cos(x^2)\cdot\cancel{2}x}\right)$

$= \lim\limits_{x\to 0}\left(\cancelto{1}{\frac{\sin x}{x}}\cdot\frac{\cos x}{\cos(x^2)}\right) = \lim\limits_{x\to 0}\left(\frac{\cos x}{\cos(x^2)}\right) = \frac{1}{1} = 1$

You try it!

b) $\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\left((1-\sin x)\cdot\tan x\right) = ?$

A) 1

B) 0

C) ∞

D) $-\infty$

Solution: $\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\left((1-\sin x)\cdot\tan x\right) \overset{\text{"DSP" " "}}{=} 0\cdot\infty$

Re-write to obtain $\frac{\infty}{\infty}$ or $\frac{0}{0}$

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-}\left(\frac{1-\sin x}{\cot x}\right) \overset{\text{"DSP"}}{=} \text{"}\frac{0}{0}\text{"} \quad \text{L.R} \checkmark$$

$$\overset{H}{=} \lim_{x\to\left(\frac{\pi}{2}\right)^-}\left(\frac{-\cos x}{-\csc^2 x}\right) = \lim_{x\to\left(\frac{\pi}{2}\right)^-}\left(\cos x\cdot\sin^2 x\right)$$

$$\left[\csc(x) = \frac{1}{\sin(x)}\right]$$

$$\overset{\text{"DSP"}}{=} 0\cdot 1 = 0$$

You try! $\lim_{x \to 0} \left(\frac{\ln(x^2+1)}{x} \right) = ?$

A) $\ln 2$

B) e^2

C) 4

D) 0

E) none of the above

$$\lim_{x\to 0}\left(\frac{\ln(x^2+1)}{x}\right) = ?$$

Solution: $$\lim_{x\to 0}\left(\frac{\ln(x^2+1)}{x}\right) \overset{\text{"DSP"}}{=} \frac{\ln(0^2+1)}{0} = \text{"}\frac{0}{0}\text{"} \text{ LR } \checkmark$$

$$\overset{H}{=} \lim_{x\to 0}\left(\frac{\frac{2x}{x^2+1}}{1}\right) = \lim_{x\to 0}\left(\frac{2x}{x^2+1}\right) \overset{\text{"DSP"}}{=} \frac{0}{0^2+1} = 0$$

Exp) $\lim_{x \to 0^+} (\sin x)^{\frac{1}{\ln(\sqrt{x})}}$

Solution: $\lim_{x \to 0^+} (\sin x)^{\frac{1}{\ln(\sqrt{x})}} \overset{\text{"DSP"}}{=} (\sin 0^+)^{\frac{1}{\ln(\sqrt{0^+})}}$

$\overset{\text{"DSP"}}{=} \text{"} 0^0 \text{"}$

Re-write it by using logarithmic differentiation

$$\ln L = \ln \lim_{x \to 0^+} \left((\sin x)^{\frac{1}{\ln(\sqrt{x})}} \right)$$

$$= \lim_{x \to 0^+} \left(\ln \left(\sin x \right)^{\frac{1}{\ln(\sqrt{x})}} \right)$$

$$= \lim_{x \to 0^+} \left(\frac{1}{\ln(\sqrt{x})} \cdot \ln(\sin x) \right) \overset{\text{"DSP"}}{=} \text{"}\frac{-\infty}{-\infty}\text{"}$$

L.R. ✓

$$\overset{H}{=} \lim_{x \to 0^+} \left(\frac{\frac{\cos x}{\sin x}}{\frac{\frac{1}{2\sqrt{x}}}{\sqrt{x}}} \right)$$

$$= \lim_{x \to 0^+} \left(\frac{\frac{\cos x}{\sin x}}{\frac{1}{2x}} \right)$$

$$= \lim_{x \to 0^+} \left(\frac{2x \cdot \cos x}{\sin x} \right) = \lim_{x \to 0^+} \left(\cancel{\frac{x}{\sin x}}^{\,1} \cdot 2 \cdot \cos x \right)$$

$$= \lim_{x \to 0^+} (2 \cdot \cos x) = 2$$

$$\ln L = 2 \Rightarrow L = e^2$$