3.7. Related Rates and Applications

Goal: To relate the rate of change of different variables (w/ respect to another variable, such as time)

Procedure:

Step 1, Draw a figure, assign variables to quantities that vary.

Step 2, Find a formula/equation that relates the variables

Step 3, Differentiate the equations (usually implicitly w/ respect to time)

Step 4, Substitute specific numerical values & solve algebraically for any required rate. List known/unknown quantities you want to solve for.

Exp1) A spherical balloon is being filled with a gas in such a way that when the radius is 2 ft, the radius is increasing at the rate of $\frac{1}{6}\frac{ft}{min}$. How fast is the volume changing at this time?

Given: when $r=2$ ft, $\frac{dr}{dt}=\frac{1}{6}\frac{ft}{min}$

question: $\frac{dV}{dt}=?$

$V \to$ volume of a sphere: $V=\frac{4}{3}\pi r^3$

$V=\frac{4}{3}\pi r^3$ (take der. w/ respect to time)

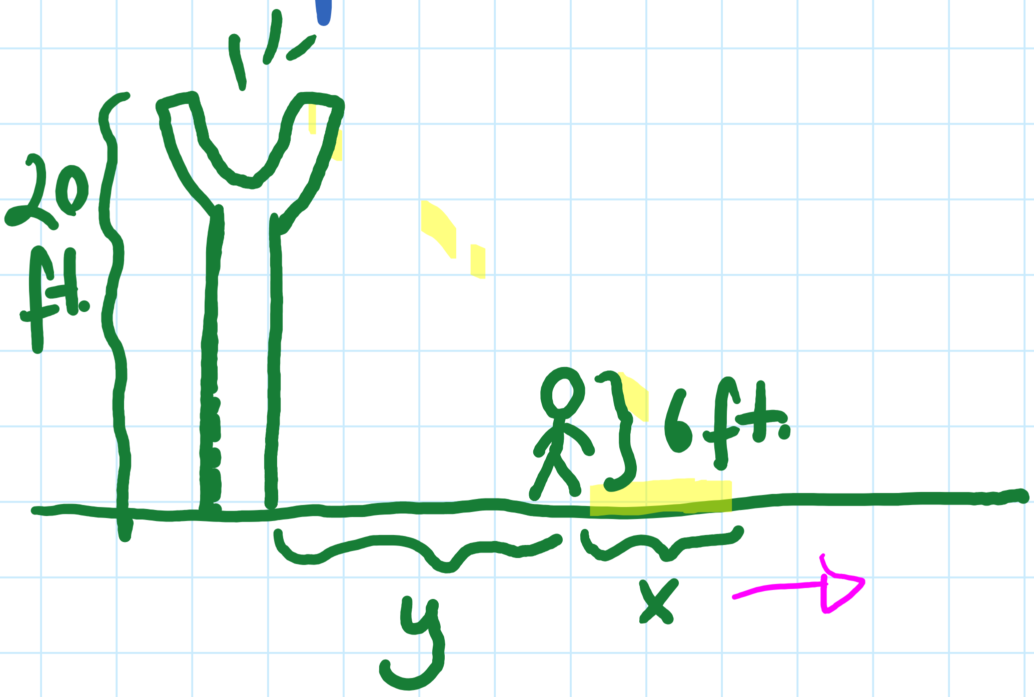
$$\frac{dV}{dt}=\frac{4}{3}\cdot\pi\cdot 3r^2\cdot\frac{dr}{dt}$$

$$=\frac{\cancel{4}^{2}}{\cancel{3}}\pi\cdot\cancel{3}\cdot 2^2\cdot\frac{1}{\cancel{6}_{3}}$$ (at that moment in time)

$$\boxed{\frac{dV}{dt}=\frac{8\pi}{3}\ \frac{ft^3}{min.}}$$

Exp2) Moving Shadow Problem

A person 6 ft. tall is walking away from a street light 20 ft. high at the rate of 7 ft/s. At what rate is the length of the person's shadow increasing?



$x \to$ length of the shadow

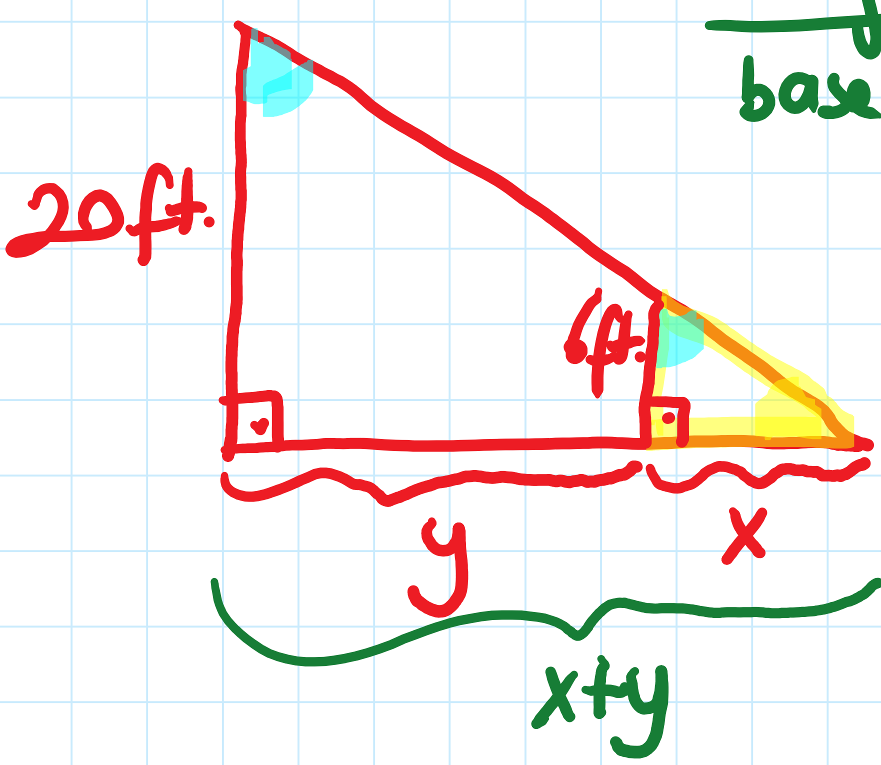
$y \to$ distance between the street light & person.

$$\frac{dy}{dt} = 7 \frac{ft}{sec.}$$

What's $\frac{dx}{dt} = ?$

Based on similar Δs:

$$\frac{\text{height of the larger } \Delta}{\text{base of the larger } \Delta} = \frac{\text{height of smaller } \Delta}{\text{base of smaller } \Delta}$$



$$\frac{20}{x+y} = \frac{6}{x}$$

$$20 \cdot x = 6(x+y)$$

$$20x = 6x + 6y$$

$$-6x \quad -6x$$

$$\boxed{14x = 6y}$$

$$14x = 6 \cdot y$$ (differentiate w/ respect to time)

$$14 \cdot \frac{dx}{dt} = 6 \cdot \frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{6}{14} \cdot \frac{dy}{dt}$$ (Recall: $\frac{dy}{dt} = 7 \text{ ft/sec.}$)

$$\frac{dx}{dt} = \frac{6}{\cancel{14}_{2}} \cdot \cancel{7} = 3$$

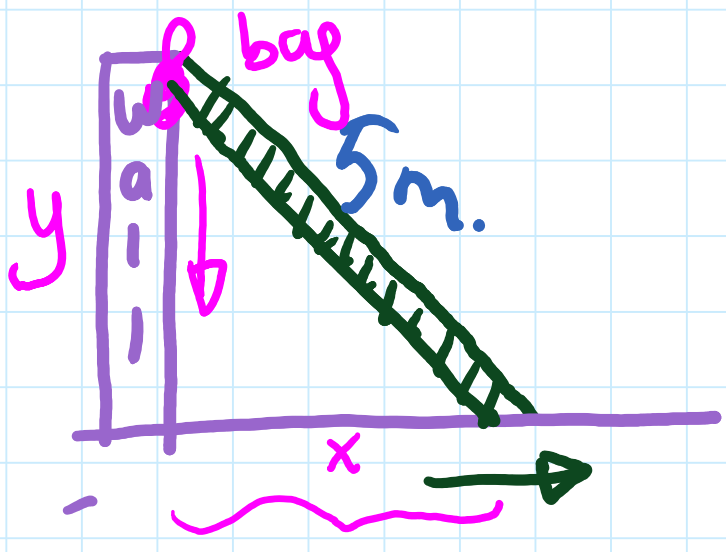
$$\frac{dx}{dt} = 3 \text{ ft/sec.}$$

The length of the person's shadow is increasing at the rate of 3 ft/sec.

Exp3) Leaning Ladder Problem

A bag is tied to the top of a 5m-ladder resting against a vertical wall. Suppose the ladder begins sliding down the wall in such a way that the foot of the ladder is moving away from the wall. How fast is the bag descending at the instant the foot of the ladder is 4m. from the wall and the foot is moving away at the rate of 2m/s?

Solution:

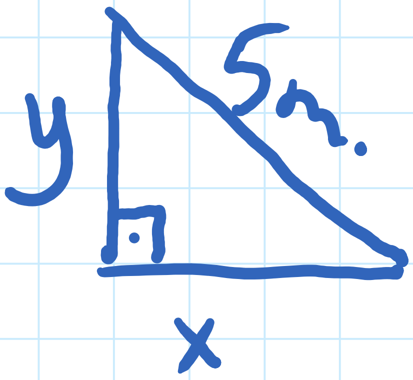


$x \to$ distance between the foot of the ladder & the base of the wall

$y \to$ distance between the bag & the base of the wall

Find $\frac{dy}{dt}$ when $x = 4m.$, $\frac{dx}{dt} = 2\frac{m}{sec}$

General Observation:



$$x^2+y^2=5^2$$ (derivative of both sides w/ respect to t)

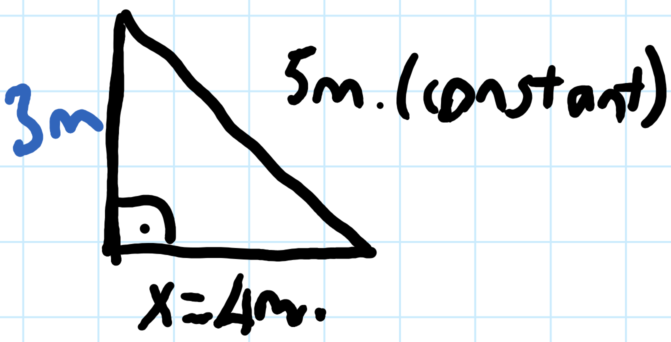
$$\cancel{2x\cdot\frac{dx}{dt}}+2y\cdot\frac{dy}{dt}=0$$

$$\cancel{-2x\cdot\frac{dx}{dt}} \qquad\qquad -2x\cdot\frac{dx}{dt}$$

$$\cancel{2}y\cdot\frac{dy}{dt}=-\cancel{2}x\cdot\frac{dx}{dt}$$

$$\frac{dy}{dt}=\frac{-x\cdot\frac{dx}{dt}}{y}$$

Specific Observation



$\left\{ \text{when } x=4m,\ y=3m \quad \frac{dx}{dt}=2m/s. \right\}$

(3-4-5 special right triangle)

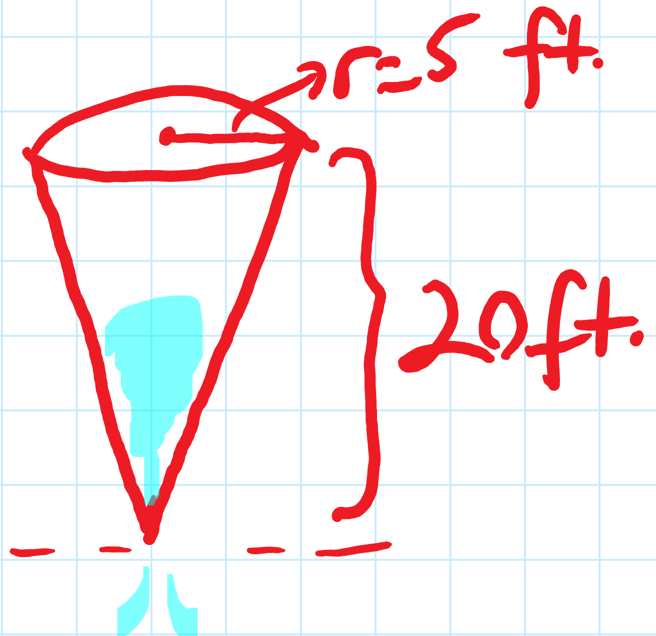
$$\frac{dy}{dt}=\frac{-x\cdot\frac{dx}{dt}}{y}$$

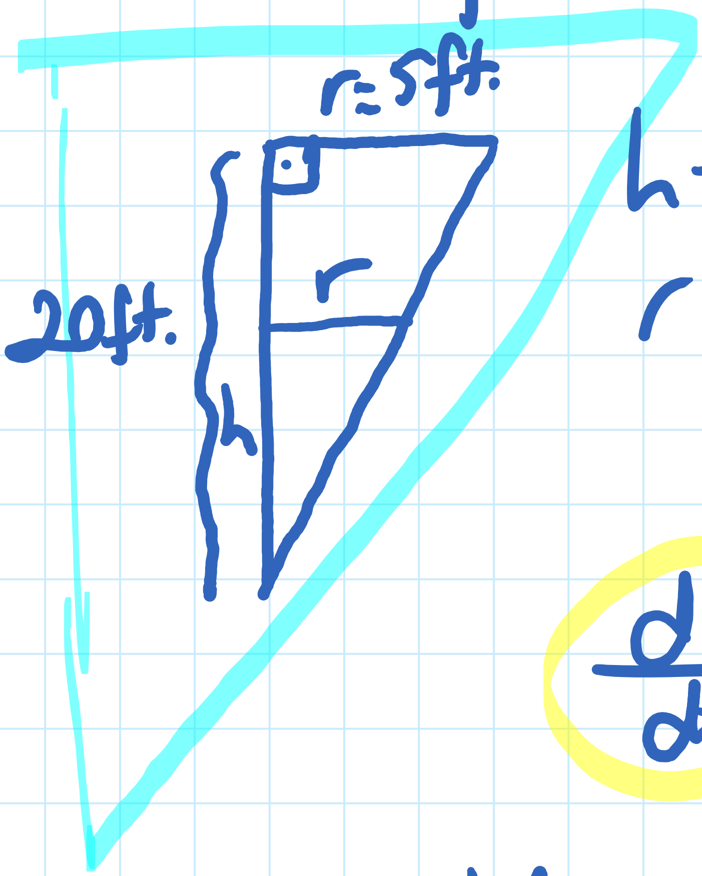
$$\frac{dy}{dt}=\frac{-4\cdot 2}{3}=-\frac{8}{3}\ \frac{m}{s}$$

The bag is descending at the rate of $\frac{8}{3}\ \frac{m}{sec}$.

Exp 4) The water level in a cone-shaped tank

A tank filled with water in the shape of an inverted cone 20 ft. high with a circular base (on top) whose radius is 5 ft. Water is running out of the bottom of the tank at the constant rate of 2 ft³/min. How fast is the water level falling when the water is 8 ft. deep?



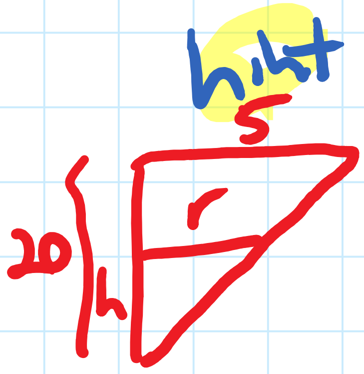


h → water level

r → radius of the cone that has water.

$$\frac{dh}{dt} = ? \text{ when } h = 8\text{ft.}$$

$$\frac{dV}{dt} = 2\frac{ft^3}{min.}$$



base : height

$$\frac{5}{20} = \frac{r}{h}$$

$$5h = 20r$$

$$\frac{h}{4} = r$$

$$V = \frac{1}{3} \cdot \pi r^2 h$$

$$V = \frac{1}{3} \cdot \pi \cdot \left(\frac{h}{4}\right)^2 \cdot h$$

$$V = \frac{\pi}{48} \cdot h^3$$

$$V = \frac{\pi}{48} \cdot h^3$$

(diff. w/ respect to time)

given: $h = 8$ ft.

$$\boxed{\frac{dV}{dt} = -2 \ \frac{ft^3}{min}}$$

$$\frac{dV}{dt} = \frac{\pi}{48} \cdot 3h^2 \cdot \frac{dh}{dt}$$

$$-2 = \frac{\pi}{\cancel{48}\ \cancel{16}\ \cancel{2}} \cdot \cancel{3} \cdot \cancel{8} \cdot \cancel{8} \cdot \frac{dh}{dt}$$

$$\frac{-2}{4\pi} = \frac{4\pi}{4\pi} \cdot \frac{dh}{dt}$$

$$\frac{-2}{4\pi} = \frac{-1}{2\pi} \frac{ft}{min} = \frac{dh}{dt}$$

The water level is decreasing at the rate of

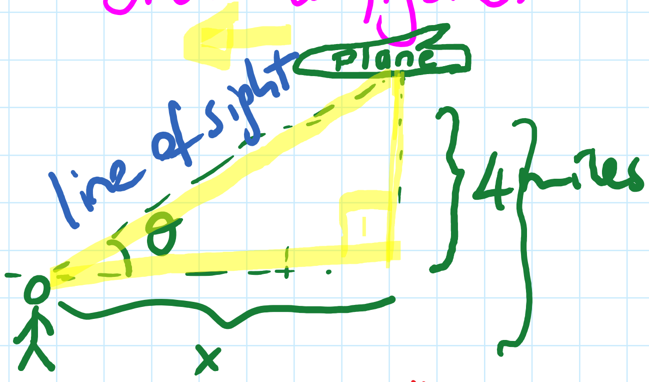
$\frac{1}{2\pi} \ \frac{ft}{min}$ $\left\{ \text{recall } \pi \approx 3.14 ;\ \frac{1}{2\pi} \approx \frac{1}{6} \frac{ft}{min} \right\}$

(1 ft. per 6 minutes)

Exp5) Angle of Elevation

Every day, a flight to L.A. flies directly over my home at a constant altitude of 4 miles. If I assume that the plane is flying at a constant speed of 400 mi/h, at what rate is the angle of elevation of my line of sight changing with respect to time when the horizontal distance between the approaching plane and my location is exactly 3 miles?

Draw a figure:



* x is decreasing *

$\frac{dx}{dt}$ is negative

Given:

$x \rightarrow$ distance between plane and person

4 miles altitude (constant)

$\theta \rightarrow$ angle of elevation

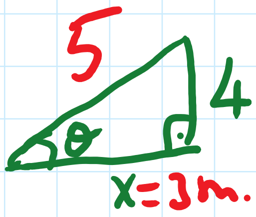
$\frac{dx}{dt} = -400$ mi/h.

Find $\frac{d\theta}{dt}$ when $x = 3$ miles. ?

$\tan\theta = \frac{4}{x} = 4\cdot x^{-1}$ (come up w/ a formula)

$\sec^2\theta \cdot \frac{d\theta}{dt} = -4\cdot x^{-2}\cdot \frac{dx}{dt}$ (take the derivative of both sides w/ respect to time)

{specific observation when $x = 3$ m.}



$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$

$\sec^2\theta \cdot \frac{d\theta}{dt} = -4\cdot x^{-2}\cdot \frac{dx}{dt}$

$\left(\frac{5}{3}\right)^2 \cdot \frac{d\theta}{dt} = +4\cdot 3^{-2}\cdot +400$

$\frac{9}{25}\cdot\cancel{\frac{25}{9}}\cdot\frac{d\theta}{dt} = \frac{4\cdot \cancelto{16}{400}}{\cancel{9}}\cdot\frac{\cancel{9}}{\cancel{25}}$

$\boxed{\frac{d\theta}{dt} = 64\,\frac{rad}{h}}$

Angle of elevation increases 64 radians per hour.