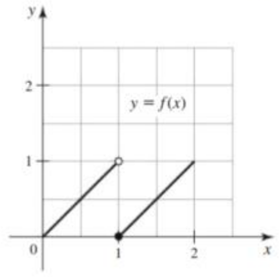
2.3 Techniques for Computing Limits

Thursday, September 10, 2020 8:35 AM

Warm -up

Use the graph of f in the figure to find the following values or state that they do not exist. If a limit does not exist, explain why.

a. $f(1)$ **b.** $\lim_{x \to 1^-} f(x)$ **c.** $\lim_{x \to 1^+} f(x)$ **d.** $\lim_{x \to 1} f(x)$



Graphical and numerical techniques for estimating limits, like those presented in the previous section, provide intuition about limits. These techniques, however, occasionally lead to incorrect results. Therefore, we turn our attention to analytical methods for evaluating limits precisely.

THEOREM 2.2 Limits of Linear Functions
Let a, b, and m be real numbers. For linear functions $f(x) = mx + b$,

$$\lim_{x \to a} f(x) = f(a) = ma + b.$$

"Direct Substitution Property" - DSP

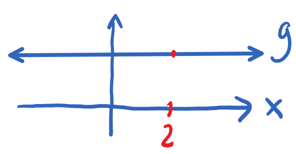
EXAMPLE 1 Limits of linear functions Evaluate the following limits.

a. $\lim_{x \to 3} f(x)$, where $f(x) = \frac{1}{2}x - 7$ **b.** $\lim_{x \to 2} g(x)$, where $g(x) = 6$

a) Use DSP

$$\lim_{x \to 3} f(x) = f(3) = \frac{1}{2} \cdot 3 - \frac{7}{1}_{(2)} = \frac{3 - 14}{2} = \frac{-11}{2}$$

b) $g(x) = 6$



constant f.

$$\lim_{x \to 2} g(x) = g(2) = 6$$

Limit Laws

Thursday, September 10, 2020 7:24 AM

THEOREM 2.3 Limit Laws

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. The following properties hold, where c is a real number, and $n > 0$ is an integer.

1. **Sum** $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. **Difference** $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. **Constant multiple** $\lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x)$
4. **Product** $\lim_{x \to a} (f(x)g(x)) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$
5. **Quotient** $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$
6. **Power** $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n$
7. **Root** $\lim_{x \to a} (f(x))^{1/n} = \left(\lim_{x \to a} f(x)\right)^{1/n}$, provided $f(x) > 0$, for x near a, if n is even

EXAMPLE 2 Evaluating limits Suppose $\lim_{x \to 2} f(x) = 4$, $\lim_{x \to 2} g(x) = 5$, and $\lim_{x \to 2} h(x) = 8$. Use the limit laws in Theorem 2.3 to compute each limit.

a. $\lim_{x \to 2} \frac{f(x) - g(x)}{h(x)}$ **b.** $\lim_{x \to 2} (6f(x)g(x) + h(x))$ **c.** $\lim_{x \to 2} (g(x))^3$

a) $\dfrac{\lim_{x \to 2} (f(x) - g(x))}{\lim_{x \to 2} h(x)} = \dfrac{\lim_{x \to 2} f(x) - \lim_{x \to 2} g(x)}{\lim_{x \to 2} h(x)} = \dfrac{4-5}{8} = \dfrac{-1}{8}$

b) $6 \cdot \lim_{x \to 2} f(x) \cdot \lim_{x \to 2} g(x) + \lim_{x \to 2} h(x) = 6 \cdot 4 \cdot 5 + 8$
$= 128$

c) $\left(\lim_{x \to 2} g(x)\right)^3 = 5^3 = 125$

Limits of Polynomials and Rational Functions

Thursday, September 10, 2020 7:25 AM

THEOREM 2.4 Limits of Polynomial and Rational Functions
Assume p and q are polynomials and a is a constant.

a. Polynomial functions: $\lim_{x \to a} p(x) = p(a)$ DSP

b. Rational functions: $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$, provided $q(a) \neq 0$ ✓

EXAMPLE 3 **Limit of a rational function** Evaluate $\lim_{x \to 2} \frac{3x^2 - 4x}{5x^3 - 36}$.

$$\lim_{x \to 2} \left(\frac{3x^2 - 4x}{5x^3 - 36} \right) \overset{DSP}{=} \frac{3 \cdot 2^2 - 4 \cdot 2}{5 \cdot 2^3 - 36} = \frac{12 - 8}{40 - 36} = \frac{4}{4} = 1$$

QUICK CHECK 2 Use Theorem 2.4 to compute $\lim_{x \to 1} \frac{5x^4 - 3x^2 + 8x - 6}{x + 1}$.

Limit of an Algebraic Function

Thursday, September 10, 2020 7:27 AM

EXAMPLE 4 **An algebraic function** Evaluate $\lim_{x \to 2} \frac{\sqrt{2x^3 + 9} + 3x - 1}{4x + 1}$.

Use DSP $\Rightarrow \dfrac{\lim_{x \to 2} \left(\sqrt{2x^3+9} + 3x - 1\right)}{\lim_{x \to 2} (4x+1)}$

$$= \frac{\sqrt{2 \cdot 2^3 + 9} + 3 \cdot 2 - 1}{4 \cdot 2 + 1} = \frac{\sqrt{25} + 6 - 1}{9} = \boxed{\frac{10}{9}}$$

Evaluate $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$

Limit Laws for One-Sided Limits

Thursday, September 10, 2020 7:30 AM

$\cancel{\sqrt{-2}}$

THEOREM 2.3 (CONTINUED) Limit Laws for One-Sided Limits

Laws 1–6 hold with $\lim\limits_{x\to a}$ replaced with $\lim\limits_{x\to a^+}$ or $\lim\limits_{x\to a^-}$. Law 7 is modified as follows. Assume $n > 0$ is an integer.

7. Root

a. $\lim\limits_{x\to a^+} (f(x))^{1/n} = \left(\lim\limits_{x\to a^+} f(x)\right)^{1/n}$, provided $f(x) \ge 0$, for x near a with $x > a$, if n is even

b. $\lim\limits_{x\to a^-} (f(x))^{1/n} = \left(\lim\limits_{x\to a^-} f(x)\right)^{1/n}$, provided $f(x) \ge 0$, for x near a with $x < a$, if n is even

$f(x) = \sqrt{x} = x^{1/2}$

$x \geq 0$

$x=1$ transition point

EXAMPLE 5 **Calculating left- and right-sided limits** Let

$x=1$

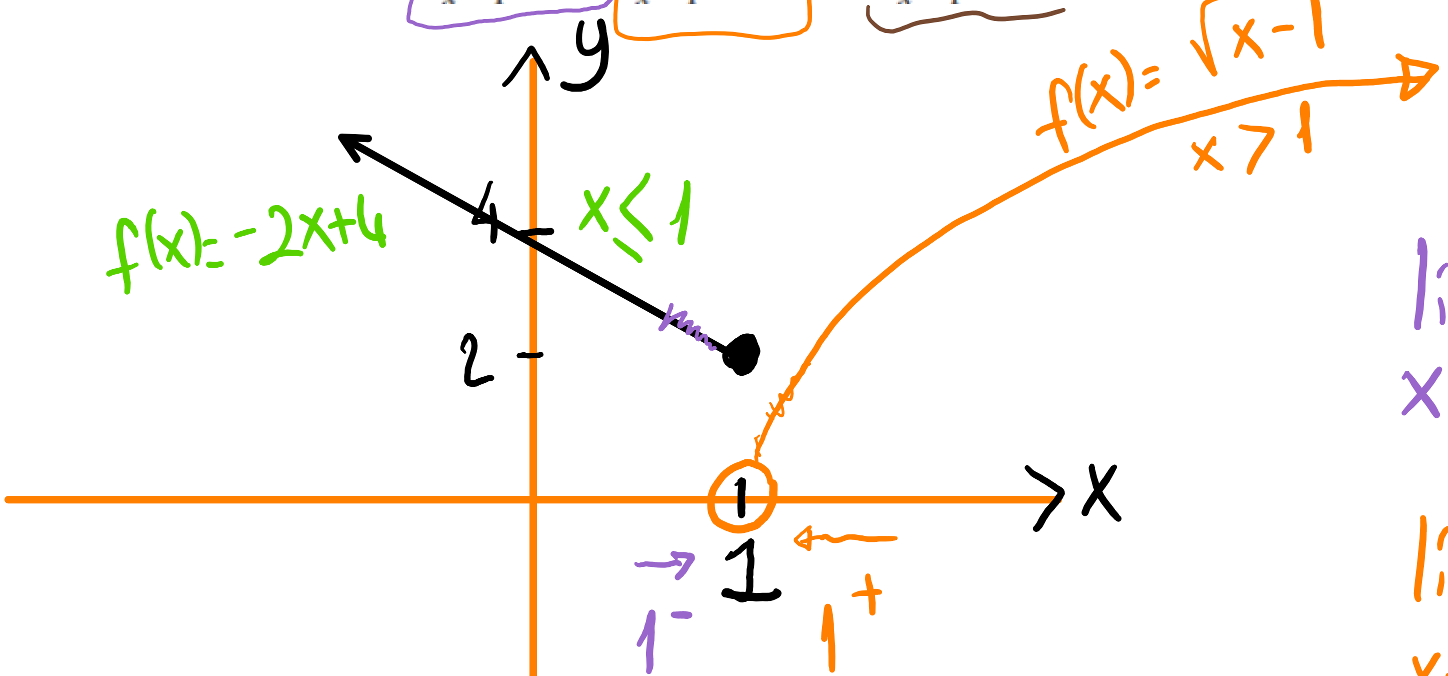
$x=1 \quad \sqrt{1-1} = 0$

$$f(x) = \begin{cases} -2x+4 & \text{if } x \le 1 \\ \sqrt{x-1} & \text{if } x > 1. \end{cases}$$

→ linear f. $y = mx+b$; $m=-2$, $b=4$

→ radical f. $y=\sqrt{x} \Rightarrow \sqrt{x-1}$

Find the values of $\lim\limits_{x\to 1^-} f(x)$, $\lim\limits_{x\to 1^+} f(x)$, and $\lim\limits_{x\to 1} f(x)$, or state that they do not exist.



$\lim\limits_{x\to 1^-} f(x) = 2$ left limit

$\lim\limits_{x\to 1^+} f(x) = 0$ right limit

Since $\lim\limits_{x\to 1^-} f(x) \neq \lim\limits_{x\to 1^+} f(x)$ $\quad \lim\limits_{x\to 1} f(x)$ DNE

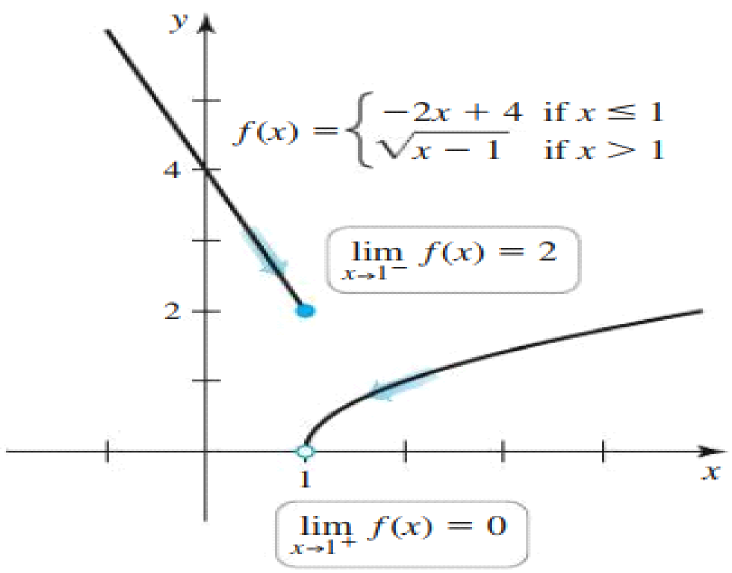


Figure 2.16

Other Techniques

Thursday, September 10, 2020 7:32 AM

Other Techniques

So far, we have evaluated limits by direct substitution. A more challenging problem is finding $\lim_{x\to a} f(x)$ when the limit exists, but $\lim_{x\to a} f(x) \neq f(a)$. Two typical cases are shown in Figure 2.17. In the first case, $f(a)$ is defined, but it is not equal to $\lim_{x\to a} f(x)$; in the second case, $f(a)$ is not defined at all. In both cases, direct substitution does not work—we need a new strategy. One way to evaluate a challenging limit is to replace it with an equivalent limit that *can* be evaluated by direct substitution. Example 6 illustrates two common scenarios.

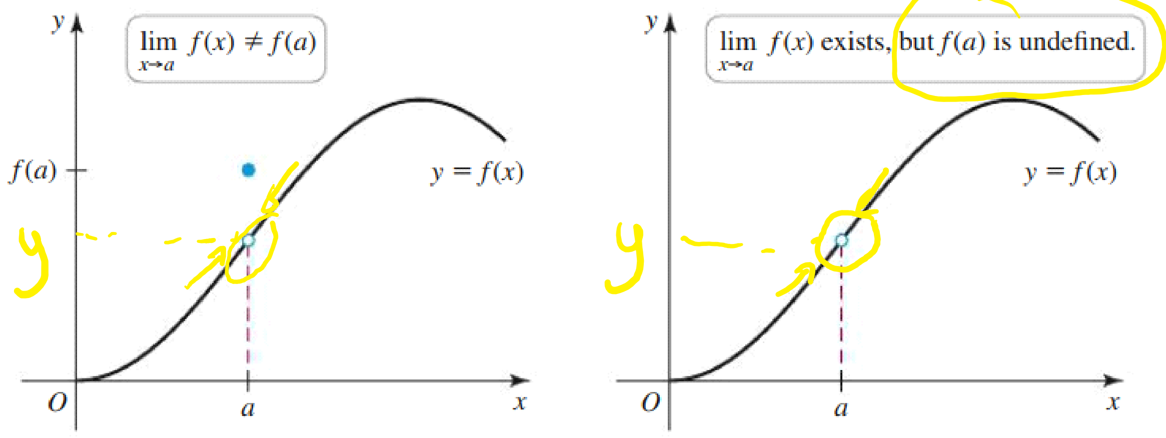


Figure 2.17

Techniques to Evaluate Limits (when DSP does not work)

a) **Factor and Cancel**

b) **Use conjugates**

Exp) Evaluate $\lim_{x\to 2}\left(\frac{x^2-6x+8}{x^2-4}\right)$

Let's try DSP : in the denominator $\Rightarrow 2^2-4=0$

$$\lim_{x\to 2}\left(\frac{(x-4)\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}\right) = \lim_{x\to 2}\left(\frac{x-4}{x+2}\right) \overset{\text{"DSP"}}{=} \frac{2-4}{2+2} = \frac{-2}{4} = \frac{-1}{2}$$

in the numerator: $\frac{2^2-6\cdot 2+8}{0} = \text{"}\frac{0}{0}\text{"}$!

Other Techniques

Thursday, September 10, 2020 7:35 AM

→ radical, try conjugate

Exp) Evaluate $\lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\right)$

Let's try DSP: $f(1) = \frac{\sqrt{1}-1}{1-1} = \boxed{\frac{0}{0}}$!

$$\lim_{x\to 1}\left(\frac{(\sqrt{x}-1)}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}\right)$$ 1

$(a-b)(a+b) = a^2 - b^2$

$(\sqrt{x}-1)(\sqrt{x}+1) = (\sqrt{x})^2 - 1^2$

$= x - 1$

$$= \lim_{x\to 1}\left(\frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x}+1)}\right) = \lim_{x\to 1}\left(\frac{1}{\sqrt{x}+1}\right)$$

$$\overset{\text{"DSP"}}{=} \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

Domain Issues

Thursday, September 10, 2020 7:35 AM

EXAMPLE 7 **Finding limits** Let $f(x) = \dfrac{x^3 - 6x^2 + 8x}{\sqrt{x-2}}$. Find the values of $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, and $\lim_{x\to 2} f(x)$, or state that they do not exist.

?

left (right)

1.999 2

for $\sqrt{x-2}$: $x-2>0 \Rightarrow x>2$

since $x>2$; $\lim_{x\to 2^-} f(x)$ does not exist

therefore; $\lim_{x\to 2} f(x)$ does not exist!

for $x>2$ (right limit)

re-write $f(x)$

$$\lim_{x\to 2^+}\left(\frac{x^3-6x^2+8x}{\sqrt{x-2}}\cdot\frac{\sqrt{x-2}}{\sqrt{x-2}}\right)$$

Try DSP

$$f(2) = \frac{2^3-6\cdot 2^2+8\cdot 2}{\sqrt{2-2}} = 0/0$$

Tip!

$$(x-4) = (\sqrt{x})^2 - 2^2 = (\sqrt{x}-2)(\sqrt{x}+2)$$

-4, -2

$$\lim_{x\to 2^+}\left(\frac{x(x^2-6x+8)\cdot\sqrt{x-2}}{(x-2)}\right)$$

$$\lim_{x\to 2^+}\left(\frac{x(x-4)\cancel{(x-2)}\sqrt{x-2}}{\cancel{x-2}}\right) = \lim_{x\to 2^+}\left(x(x-4)\sqrt{x-2}\right)$$

DSP

$$= 2(2-4)\underbrace{\sqrt{2-2}}_{0} = 0$$

Note: When DSP, if you obtain "$\frac{0}{0}$" Indeterminate Form

Use "other methods" to eval. limits.

Limits of Piece-wise Functions

left right

$\lim_{x \to -1} g(x)$, where $g(x) = \begin{cases} \dfrac{x^2 - 1}{x + 1} & \text{if } x < -1 \\ -2 & \text{if } x \geq -1 \end{cases}$

left limit

-1

$x < -1$ $\quad x \geq -1$

If LL = RL the two-sided limit exist.

DSP

$$\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} \left(\frac{x^2 - 1}{x + 1} \right) = \lim_{x \to -1^-} \frac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}} = -1-1 = -2$$

If we try DSP "$\frac{0}{0}$"

$$\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} (-2) = -2$$

Since $\lim_{x \to -1^+} g(x) = \lim_{x \to -1^-} g(x) = -2$; $\lim_{x \to -1} g(x) = -2$

Find the value that makes a limit exist

Thursday, September 10, 2020 7:41 AM

Given: $\lim_{x \to -1} g(x)$ Exist

Finding a constant Suppose

$$g(x) = \begin{cases} x^2 - 5x & \text{if } x \leq -1 \\ ax^3 - 7 & \text{if } x > -1. \end{cases}$$

Determine a value of the constant a for which $\lim_{x \to -1} g(x)$ exists and state the value of the limit, if possible.

Q: find a

$\lim_{x \to -1} g(x) = ?$

$$\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} (x^2 - 5x) \overset{DSP}{=} f(-1) = (-1)^2 - 5(-1) = 1 + 5 = 6$$

$$\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} (ax^3 - 7) = f(-1) = a(-1)^3 - 7$$

$$= -a - 7$$

Since $\lim_{x \to -1} g(x)$ exists: $\lim_{x \to -1^-} g(x) = \lim_{x \to -1^+} g(x)$

$$6 = -a - 7$$

$$\boxed{a = -13}$$

$$\lim_{x \to -1} g(x) = 6$$