3.7 The Chain Rule

The differentiation rules presented so far allow us to find derivatives of many functions. However, these rules are inadequate for finding the derivatives of most *composite functions*. Here is a typical situation. If $f(x) = x^3$ and $g(x) = 5x + 4$, then their composition is $f(g(x)) = (5x + 4)^3$. One way to find the derivative is by expanding $(5x + 4)^3$ and differentiating the resulting polynomial. Unfortunately, this strategy becomes prohibitive for functions such as $(5x + 4)^{100}$. We need a better approach.

$((\cos x))^3$

$f(x) = x^3$ $\quad (x^3)' = 3 \cdot x^2$

$f(g(x)) = (5x+4)^3$ $\quad ((5x+4)^3)'$

$(5x+4)^{100}$

THEOREM 3.13 The Chain Rule
Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1}$$

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x) \tag{2}$$

prime (Lagrange) Notation $\quad [f(g(x))]' = \underbrace{f'(g(x))}_{\text{derivative of outside function w/ respect to } g(x)} \underbrace{g'(x)}_{\text{derivative of inside function w/ respect to } x}$

$g \to$ inside f.

$f \to$ outside f.

Leibniz Notation $\quad \frac{d}{dx}[f(g(x))] = \frac{df}{dg} \cdot \frac{dg}{dx}$

$g'(x)$

$f'(g(x))$

prime (Lagrange) Notation $[f(g(x))]' = \underbrace{f'(g(x))}_{\text{derivative of outside function w/ respect to } g(x)} \cdot \underbrace{g'(x)}_{\text{derivative of inside function w/ respect to } x}$

Leibniz Notation $\frac{d}{dx}[f(g(x))] = \frac{df}{dg} \cdot \frac{dg}{dx}$

EXAMPLE 1 **The Chain Rule** Use the chain rule to find the derivative.

a. $y = (5x + 4)^{3/2}$ **b.** $y = \sin^3 x$ **c.** $y = \sin x^3$

a. $y = (\underline{5x+4})^{3/2}$

g (inside f.) $\to 5x+4$

f (outside f.) $\to x^{3/2}$

$f(g(x)) = (5x+4)^{3/2}$

$$\left[(5x+4)^{3/2}\right]' = \frac{d}{dx}\left((5x+4)^{3/2}\right)$$

$$= \underbrace{\frac{3}{2}\cdot(5x+4)^{1/2}}_{\text{power rule}} \cdot \underbrace{\frac{d(5x+4)}{dx}}_{\text{chain rule}}$$

$$= \frac{3}{2}\cdot\sqrt{5x+4}\cdot 5 = \frac{15}{2}\cdot\sqrt{5x+4}$$

Calculate the derivative:

a. $y = (5x + 4)^{3/2}$ **b.** $y = \sin^3 x$ **c.** $y = \sin x^3$ $= \sin(x^3)$

b. $y = (\sin x)^3$

$g \to \sin x$ (inside f.)

$g'(x) = \cos x$

$f \to x^3$ (outside f.)

$f(g(x)) = (\sin x)^3$

$$\frac{d}{dx}\left((\sin x)^3\right) = 3\cdot(\sin x)^2 \cdot \cos x$$

c. $y = \sin(x^3)$

$g \to x^3$ (inside f.)

$g' = 3x^2$

$f \to \sin(x)$ (outside f.)

$f(g(x)) = \sin(x^3)$

$$(\sin(x^3))' = \cos(x^3)\cdot 3x^2$$

$(\sin x)^3$ vs. $\sin(x^3)$

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x) \qquad (2)$$

THEOREM 3.14 Chain Rule for Powers
If g is differentiable for all x in its domain and p is a real number, then

$$\frac{d}{dx}\left((g(x))^p\right) = p(g(x))^{p-1}\underline{g'(x)}. \leftarrow$$

EXAMPLE 2 The Chain Rule Use equation (2) of the Chain Rule to calculate the derivatives of the following functions.

a. $(6x^3 + 3x + 1)^{10}$ **b.** $\sqrt{5x^2 + 1}$ **c.** $\left(\frac{5t^2}{3t^2 + 2}\right)^3$ **d.** e^{-3x}

a. $\frac{d}{dx}\left((\underbrace{6x^3+3x+1}_{g(x)\text{ inside f.}})^{10}\right) = 10\cdot(6x^3+3x+1)^9\cdot(6x^3+3x+1)'$

$= 10\cdot(6x^3+3x+1)^9\cdot(18x^2+3)$

b. $\frac{d}{dx}\left((5x^2+1)^{1/2}\right) = \frac{1}{2}\cdot(5x^2+1)^{-1/2}\cdot 10x$

$= 5x(5x^2+1)^{-1/2} = \frac{5x}{\sqrt{5x^2+1}}$

Recall:

$\left(\frac{f}{g}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2}$

c. $\frac{d}{dt}\left(\underbrace{\left(\frac{5t^2}{3t^2+2}\right)}_{g(t)}^3\right) = 3\cdot\left(\frac{5t^2}{3t^2+2}\right)^2\cdot\underbrace{\frac{d}{dt}\left(\frac{5t^2}{3t^2+2}\right)}_{g'(t)}$

$= 3\cdot\left(\frac{5t^2}{3t^2+2}\right)^2\cdot\underbrace{\left(\frac{10t\cdot(3t^2+2)-5t^2\cdot 6t}{(3t^2+2)^2}\right)}_{\text{quotient rule result}}$

$\cancel{30t^3}+20t-\cancel{30t^3}$

$$= 3\cdot\left(\frac{5t^2}{3t^2+2}\right)^2\cdot\underbrace{\left(\frac{10t\cdot(3t^2+2)-5t^2\cdot 6t}{(3t^2+2)^2}\right)}_{\text{quotient rule result}}$$

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$$= 3\cdot\left(\frac{5t^2}{3t^2+2}\right)^2\cdot\frac{20t}{(3t^2+2)^2} = 3\cdot\frac{25t^4}{(3t^2+2)^2}\cdot\frac{20t}{(3t^2+2)^2}$$

$$= \frac{1500t^5}{(3t^2+2)^4}$$

d) $y = e^{-3x}$

$$\frac{dy}{dx} = \left(e^{-3x}\right)'$$

$$= e^{-3x}\cdot(-3)$$

$$= -3\cdot e^{-3x}$$

$$(e^x)' = e^x\cdot 1$$

$$\boxed{(e^u)' = e^u\cdot u'}$$

E.g.

$$(e^{2x})' = e^{2x}\cdot(2x)'$$

$$= e^{2x}\cdot 2$$

$$= 2\cdot e^{2x}$$

Table 3.3

x	$f'(x)$	$g(x)$	$g'(x)$
1	5	2	3
2	7	1	4

EXAMPLE 3 **Calculating derivatives at a point** Let $h(x) = f(g(x))$. Use the values in Table 3.3 to calculate $h'(1)$ and $h'(2)$.

prime (Lagrange) Notation $[f(g(x))]' = \underbrace{f'(g(x))}_{\text{derivative of outside function w/ respect to } g(x)} \cdot \underbrace{g'(x)}_{\text{derivative of inside function w/ respect to } x}$ →

Leibniz Notation $\frac{d}{dx}[f(g(x))] = \frac{df}{dg} \cdot \frac{dg}{dx}$

$$h(x) = f(g(x))$$

$$h'(x) = f'(g(x)) \cdot g'(x)$$

$$h'(1) = f'(g(1)) \cdot g'(1)$$

$$\left(\begin{array}{l} g(1) = 2 \\ g'(1) = 3 \end{array}\right)$$

$$= f'(2) \cdot 3$$

$$= 7 \cdot 3 = 21$$

$$h'(2) = f'(g(2)) \cdot g'(2) = f'(1) \cdot g'(2) = 5 \cdot 4 = 20$$

$(e^u)' = e^u \cdot u'$

The Composition of Three or More Functions

We can differentiate the composition of three or more functions by applying the Chain Rule repeatedly, as shown in the following example.

EXAMPLE 6 **Composition of three functions** Calculate the derivative of $\sin(e^{\cos x})$.

inside of inside

$$\frac{d}{dx}\left(\sin\left(e^{\cos x}\right)\right)$$

inside

outside

$e^x \rightarrow e^{\cos x}$

$$= \cos\left(e^{\cos x}\right) \cdot e^{\cos x} \cdot (-\sin x) = -\cos\left(e^{\cos x}\right) \cdot e^{\cos x} \cdot \sin x$$

do not combine

$\cancel{\cos\left(e^{\cos x}\right)^2}$

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product
power
chain

EXAMPLE 7 **Combining rules** Find $\frac{d}{dx}\left(x^2\sqrt{x^2+1}\right)$.

$$\boxed{\text{Recall: } (f\cdot g)' = f'\cdot g + f\cdot g'}$$

$$\frac{d}{dx}\left(x^2\cdot(x^2+1)^{1/2}\right) = 2x\cdot(x^2+1)^{1/2} + x^2\cdot\left((x^2+1)^{1/2}\right)'$$

$$= 2x\cdot(x^2+1)^{1/2} + x^2\cdot\frac{1}{\cancel{2}}\cdot(x^2+1)^{-1/2}\cdot\cancel{2}x$$

$$= \frac{2x\cdot\sqrt{x^2+1}}{1}\left(\sqrt{x^2+1}\right) + \frac{x^3}{\sqrt{x^2+1}}$$

$$= \frac{\overbrace{2x\cdot(x^2+1)}^{2x^3+2x}+x^3}{\sqrt{x^2+1}} = \frac{3x^3+2x}{\sqrt{x^2+1}}$$