3.8 Implicit Differentiation

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$y = f(x)$

y is defined explicitly as a function of x

E.g.: eq. of a circle on the center w/ r

$x^2+y^2=r^2 \quad r=1$

Implicit $x^2+y^2=1^2$

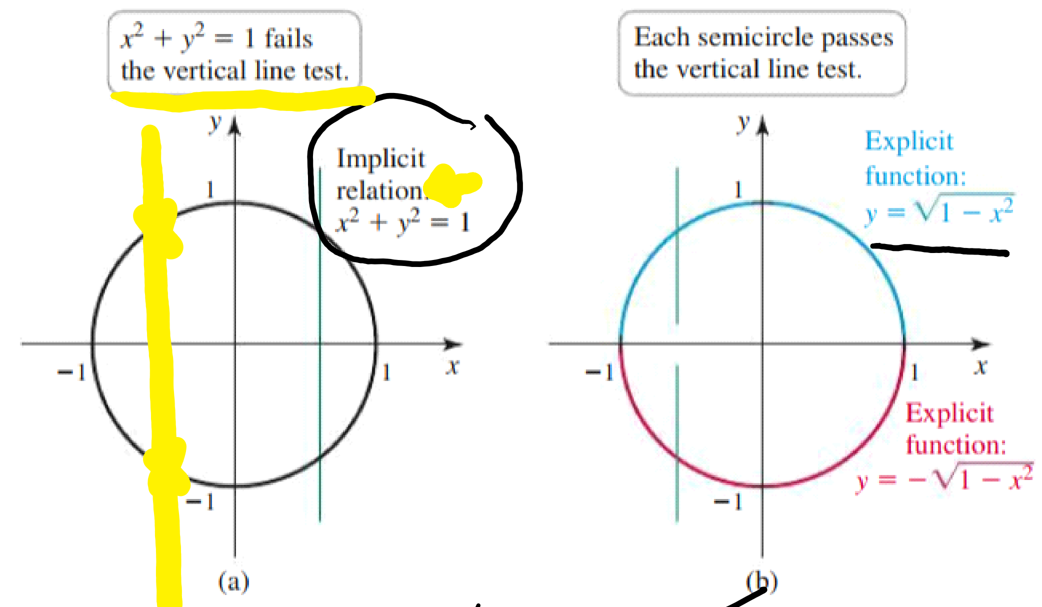
Explicit (solve for y)

$x^2+y^2=1 \Rightarrow y^2=1-x^2$

$y=\pm\sqrt{1-x^2}$

$y=\sqrt{1-x^2}$ $\qquad$ $y=-\sqrt{1-x^2}$

$\frac{dy}{dx}=-\frac{x}{\sqrt{1-x^2}}$ $\qquad$ $\frac{dy}{dx}=\frac{x}{\sqrt{1-x^2}}$



Solve for y?!

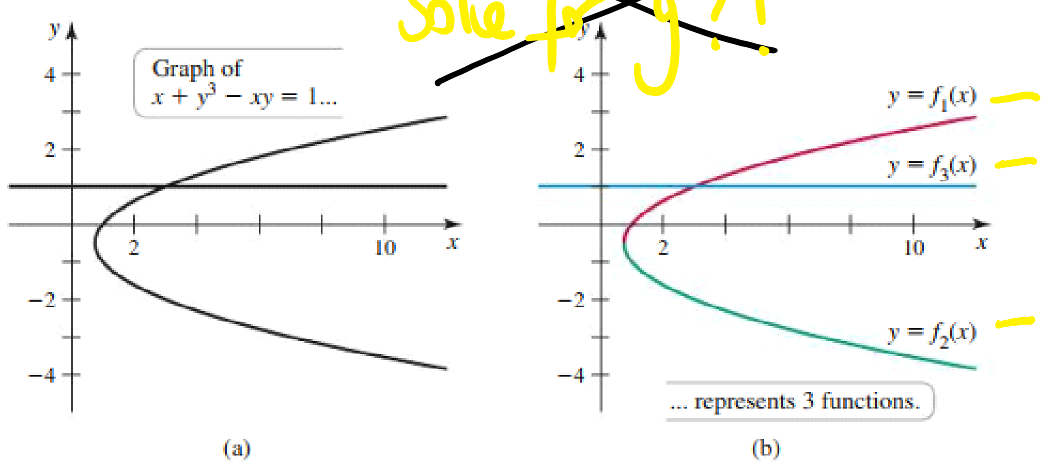


Figure 3.58

Goal: To find a single exp. for the derivative directly from an eq. w/out solving for y first.

Procedure for Implicit Differentiation

1. Differentiate both sides of an equation with respect to x. Use chain rule when differentiating terms that contain y.
2. Solve the equation algebraically for $\frac{dy}{dx}$

$\frac{dy}{dx}$ "derivative of y with respect to x"

y is a function of x

$(y(x))' = \frac{dy}{dx}$

Exp1) Find the slope of the tangent line to the circle $x^2+y^2=10$ at $P(-1,3)$

$$x^2+y^2=10$$

Differentiate both sides wrt "x"

"y is a f. of x"

$$2x+2y\cdot\frac{dy}{dx}=0$$

method 1: subs. $(-1,3)$ (x,y)

$$-2+6\cdot\frac{dy}{dx}=0$$

$$+2 \qquad +2$$

$$\frac{6\cdot\frac{dy}{dx}}{6}=\frac{2}{6}$$

$$\boxed{\frac{dy}{dx}=\frac{1}{3}}$$

#1

method 2: solve for $\frac{dy}{dx}$ first

$$2x+2y\cdot\frac{dy}{dx}=0$$

#2

$$\frac{\cancel{2y}\cdot\frac{dy}{dx}}{\cancel{2y}}=\frac{-2x}{2y}$$

$$\frac{dy}{dx}=\frac{-2x}{2y}=\frac{-x}{y}$$

subs. $(-1,3)$ / eval. at $(-1,3)$

$$\left.\frac{dy}{dx}\right|_{(-1,3)}=\frac{-(-1)}{3}=\boxed{\frac{1}{3}}$$

EXAMPLE 2 **Implicit differentiation** Find $y'(x)$ when $\sin(xy) = x^2 + y$.

No point given!

$$\sin(x\cdot y) = x^2 + y$$

"y is a f. of x"

Diff. both sides wrt x

$$\underbrace{\cos(x\cdot y)}_{\text{der. of outside f.}} \cdot \underbrace{(x\cdot y)'}_{\text{der. of inside f.}} = 2x + \frac{dy}{dx}$$

Recall:

$$(f\cdot g)' = f'g + f\cdot g'$$

$$\cos(x\cdot y)\cdot\left(1\cdot y + x\cdot\frac{dy}{dx}\right) = 2x + \frac{dy}{dx}$$

Note:

$\cos(x\cdot y)\cdot y \neq \cos(x\cdot y^2)$

$$y\cdot\cos(xy) + x\cdot\cos(xy)\cdot\frac{dy}{dx} = 2x + \frac{dy}{dx}$$

$$x\cdot\cos(xy)\cdot\frac{dy}{dx} - \frac{dy}{dx} = 2x - y\cdot\cos(xy)$$

$$\frac{dy}{dx}\left(x\cdot\cos(xy) - 1\right) = 2x - y\cdot\cos(xy)$$

$$\frac{}{(x\cdot\cos(xy) - 1)} \qquad \frac{}{(x\cdot\cos(xy) - 1)}$$

$$\frac{dy}{dx} = \frac{2x - y\cdot\cos(xy)}{x\cdot\cos(xy) - 1}$$

Slopes of Tangent Lines

Derivatives obtained by implicit differentiation typically depend on x and y. Therefore, the slope of a curve at a particular point (a, b) requires both the x- and y-coordinates of the point. These coordinates are also needed to find an equation of the tangent line at that point.

EXAMPLE 3 **Finding tangent lines with implicit functions** Find an equation of the line tangent to the curve $x^2 + xy - y^3 = 7$ at $(3, 2)$.

$$x^2+xy-y^3=7$$

"y is a f. of x"

Diff. both sides w/t x

$$2x+\left(1\cdot y+x\cdot\frac{dy}{dx}\right)-3y^2\cdot\frac{dy}{dx}=0$$

$P(3,2)$

(x,y)

Subs. $(3,2) \to (x,y)$ AFTER diff.

$$2\cdot3+\left(2+3\cdot\underline{\frac{dy}{dx}}\right)-3\cdot2^2\cdot\underline{\frac{dy}{dx}}=0$$

$\cancel{\frac{d2}{dx}}$

$2 \to \frac{dy}{dx}$

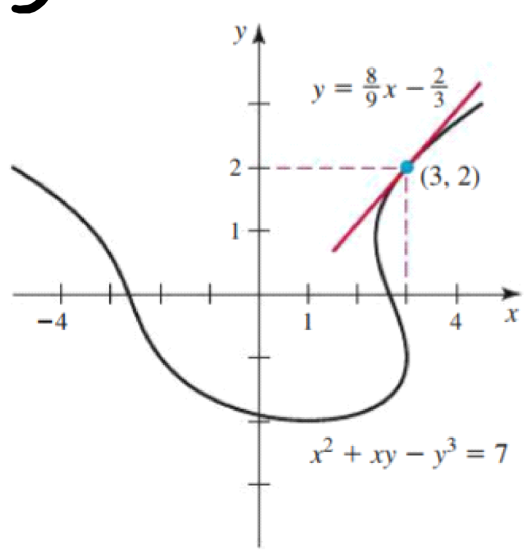
at $x=3$, $y=2$

$$\underbrace{6+2+3\frac{dy}{dx}-12\cdot\frac{dy}{dx}}=0$$

$$\frac{8}{9}=\frac{9\cdot\frac{dy}{dx}}{9} \Leftarrow 8-9\cdot\frac{dy}{dx}=0 \Rightarrow \boxed{\frac{dy}{dx}=\frac{8}{9}}$$

eq. of the tan. line at $(3,2)$ to the curve:

$$(3,2),\ \frac{dy}{dx}=\frac{8}{9} \qquad y-2=\frac{8}{9}(x-3)$$



QUICK CHECK 3 If a function is defined explicitly in the form $y = f(x)$, which coordinates are needed to find the slope of a tangent line—the x-coordinate, the y-coordinate, or both? ◀

EXAMPLE 4 **Slope of a curve** Find the slope of the curve $2(x+y)^{1/3} = y$ at the point $(4, 4)$.

"y is a f. of x"

$$2(x+y)^{1/3} = y$$

Diff. both sides wrt $\underline{x}$

$$2 \cdot \frac{1}{3}(x+y)^{-2/3} \cdot [(x+y)]' = \frac{dy}{dx} = y'$$

$$\frac{2}{3}(x+y)^{-2/3} \cdot \left(1+\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$P(4,4)$

Subs. (4,4) for (x,y)

$$\left(8^{-2/3} = (2^3)^{-\frac{2}{3}}\right)$$

$$\frac{2}{3}(4+4)^{-2/3} \cdot \left(1+\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$2^{\cancel{3} \cdot \frac{-2}{\cancel{3}}} = 2^{-2}$$

$$\frac{2}{3} \cdot (2^3)^{-2/3}\left(1+\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{1}{4} = \frac{1}{2^2}$$

$$\frac{2}{3} \cdot \cancel{2^{-2}} \cdot \left(1+\frac{dy}{dx}\right) = \frac{dy}{dx} \qquad \frac{1}{4}$$

$$\left(\frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{4}_{2}} = \frac{1}{6}\right)$$

$$\frac{1}{6}\left(1+\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{6}{6}x - \frac{1}{6}x \qquad \frac{5x}{6}$$

$$\frac{1}{6} + \underbrace{\frac{1}{6} \cdot \frac{dy}{dx}}_{} = \frac{dy}{dx} \Rightarrow \frac{1}{6} = \frac{6}{6}\frac{dy}{dx} - \frac{1}{6} \cdot \frac{dy}{dx} = \frac{5}{6} \cdot \frac{dy}{dx}$$

$$\frac{1}{6} = \frac{5}{6} \cdot \frac{dy}{dx}$$

$$\boxed{\frac{1}{5} = \frac{dy}{dx}}$$

Higher-Order Derivatives of Implicit Functions

In previous sections of this chapter, we found higher-order derivatives $\frac{d^n y}{dx^n}$ by first calculating $\frac{dy}{dx}, \frac{d^2 y}{dx^2}, \ldots,$ and $\frac{d^{n-1} y}{dx^{n-1}}$. The same approach is used with implicit differentiation.

EXAMPLE 5 **A second derivative** Find $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 1$.

$$\frac{dy}{dx} \rightarrow \quad \cancel{2x} + 2y \cdot \frac{dy}{dx} = 0$$

$$\cancel{-2x} \qquad\qquad -2x$$

$$\frac{\cancel{2y} \cdot \frac{dy}{dx}}{\cancel{2y}} = \frac{-\cancel{2}x}{\cancel{2}y}$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

$$\frac{d^2y}{dx^2} = ? \quad \text{diff. w/t } x$$

$$\frac{d}{dx}\left(\frac{-x}{y}\right) = \frac{(-x)' \cdot y + (+x) \cdot \frac{dy}{dx}}{y^2} = \frac{-1 \cdot y + x \cdot \frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-y + x \cdot \left(\frac{-x}{y}\right)}{y^2} = \frac{\frac{-y}{1}\,(y) - \frac{x^2}{y}}{y^2} = \frac{\frac{-y^2 - x^2}{y}}{\frac{y^2}{1}}$$

$$\frac{d^2y}{dx^2} = \frac{-y^2 - x^2}{y^3}$$

given: $x^2 + y^2 = 1$

$$-\frac{1}{y^3}$$

Identify the Error

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Identify and correct the error in the following argument. Suppose $y^2 + 2y = 2x^3 - 7$. Differentiating both sides with respect to x to find $\frac{dy}{dx}$, we have $2y + 2\frac{dy}{dx} = 6x^2$, which implies that $\frac{dy}{dx} = 3x^2 - y$.