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Increasing and Decreasing Functions

We have used the terms *increasing* and *decreasing* informally in earlier sections to describe a function or its graph. For example, the graph in Figure 4.20a rises as x increases, so the corresponding function is increasing. In Figure 4.20b, the graph falls as x increases, so the corresponding function is decreasing. The following definition makes these ideas precise.

DEFINITION Increasing and Decreasing Functions

Suppose a function f is defined on an interval I. We say that f is **increasing** on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$. We say that f is **decreasing** on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.

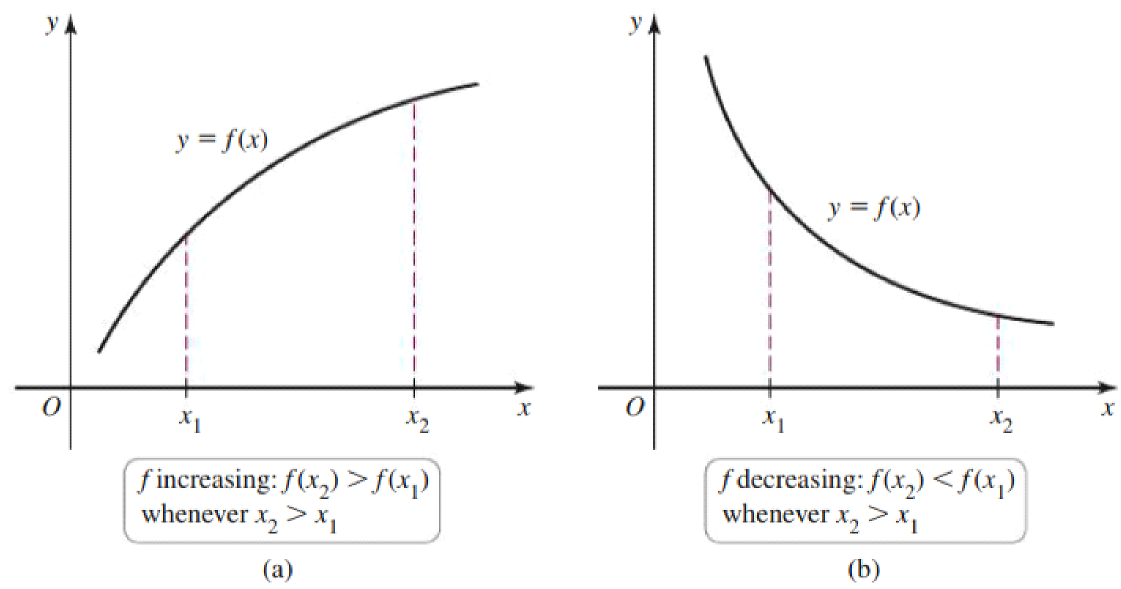


Figure 4.20

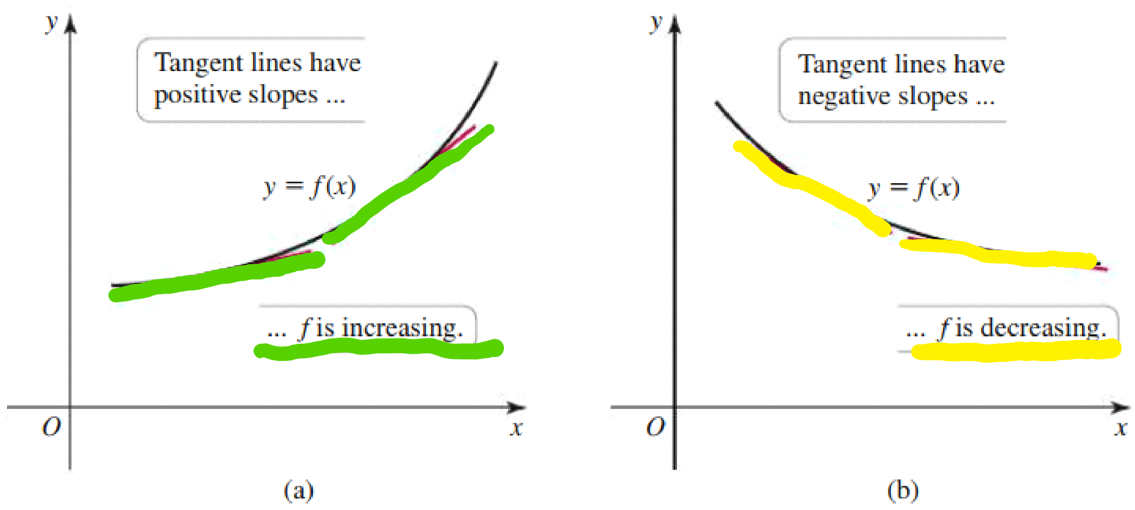
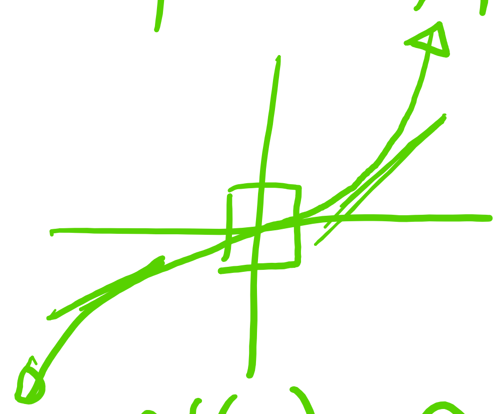


Figure 4.21

e.g.:

$f(x) = x^3$; $f'(x) = 3x^2$



$f'(0) = 0$

THEOREM 4.7 Test for Intervals of Increase and Decrease

Suppose f is continuous on an interval I and differentiable at all interior points of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

➤ The converse of Theorem 4.7 may not be true. According to the definition, $f(x) = x^3$ is increasing on $(-\infty, \infty)$ but it is not true that $f'(x) > 0$ on $(-\infty, \infty)$ (because $f'(0) = 0$).

Derivative of a function gives the slope of a tangent line

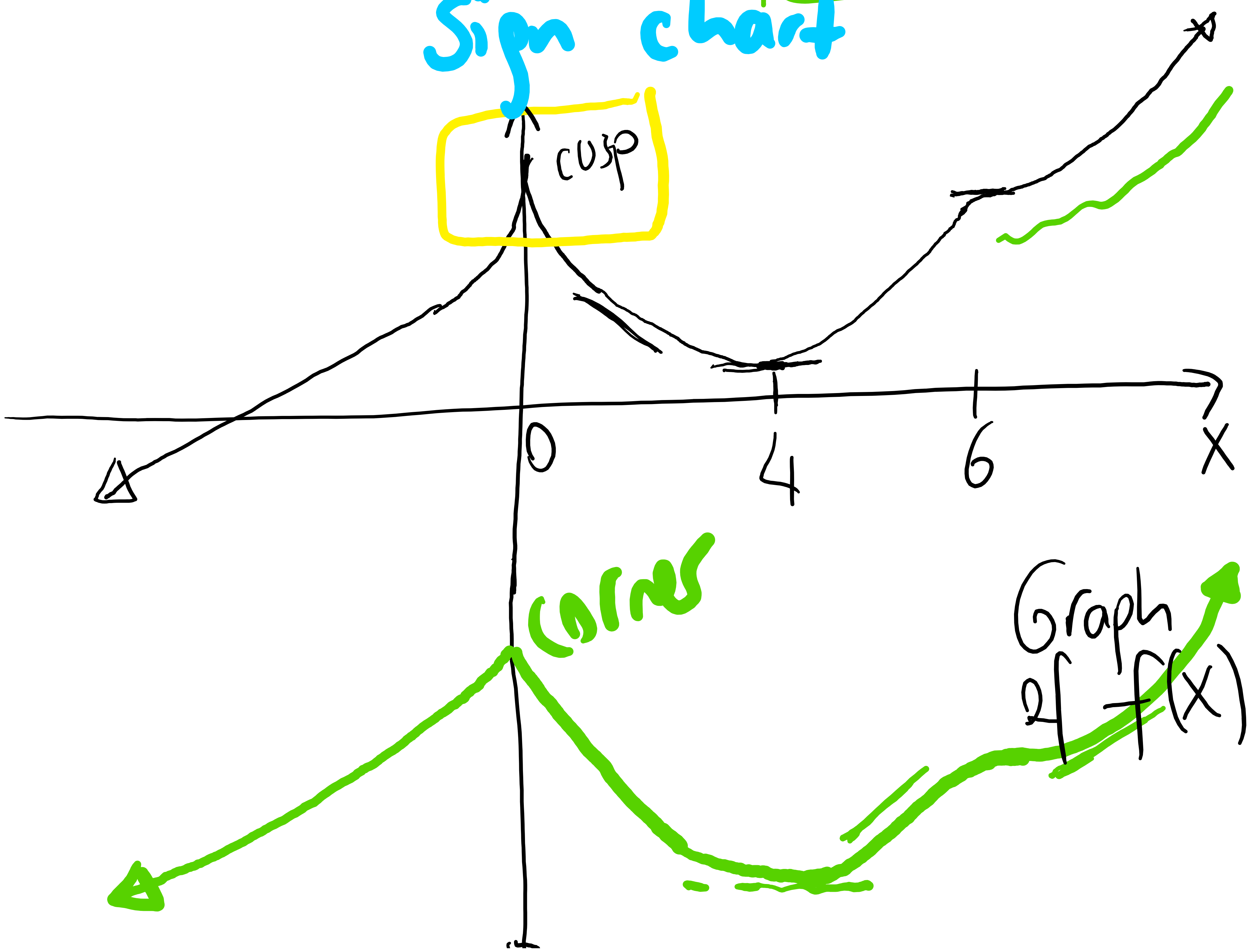
If derivative (f') is positive. The $m_{tan} > 0$ on I.
(f is increasing)

EXAMPLE 1 **Sketching a function** Sketch a graph of a function f that is continuous on $(-\infty, \infty)$ and satisfies the following conditions.

1. $f' > 0$ on $(-\infty, 0)$, $(4, 6)$, and $(6, \infty)$.
2. $f' < 0$ on $(0, 4)$.
3. $f'(0)$ is undefined.
4. $f'(4) = f'(6) = 0$.

	$-\infty$		0		4		6		$+\infty$
Sign of $f'(x)$		pos.	undf.	neg.	0	pos.	0	pos.	
behavior of $f(x)$		incr.	cusp corner	decr.	hor. tag.	incr.	hor. tag.	incr.	

Sign chart



EXAMPLE 2 **Intervals of increase and decrease** Find the intervals on which the following functions are increasing and the intervals on which they are decreasing.

a. $f(x) = xe^{-x} = x \cdot e^{-x}$

$$f'(x) = 1 \cdot e^{-x} + x \cdot e^{-x} \cdot (-1) = e^{-x}(1-x) = \frac{1-x}{e^x}$$

critical p. $f'(x) = 0$ or DNE

$$f'(x) = 0 = \frac{1-x}{e^x} \Rightarrow 1 - x = 0 \Rightarrow \boxed{x = 1}$$

$f'(x)$ DNE $\to$ $e^x = 0$ none!

SIGN CHART

	$-\infty$	1	$+\infty$
sign of $f'(x)$	+	0	−
$f(x)$	incr.		decr.

$f'(x) = \frac{1-x}{e^x}$

test points: $x = 0$, $x = 2$

$$\underbrace{f'(0) = \frac{1-0}{e^0} > 0}_{f \text{ is incr.}}$$

$$\underbrace{f'(2) = \frac{1-2}{e^2} \boxed{\frac{\ominus}{\oplus}} < 0}_{f \text{ is decr.}}$$

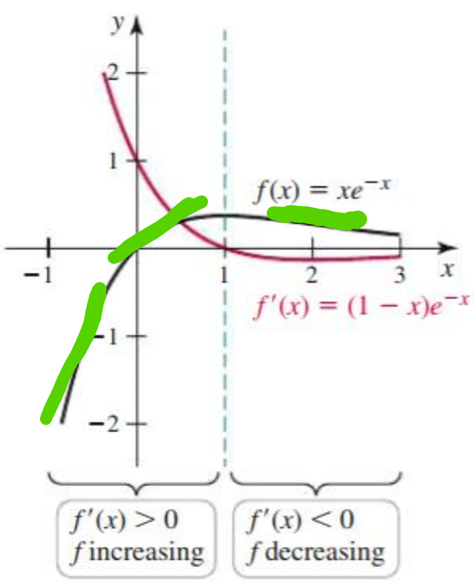


Figure 4.24

* f is incr. on $(-\infty, 1)$
* f is decr. on $(1, \infty)$
* f has a horizontal tangent at $x = 1$

Ch 4 -

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Identifying Local Maxima and Minima

[a,b]

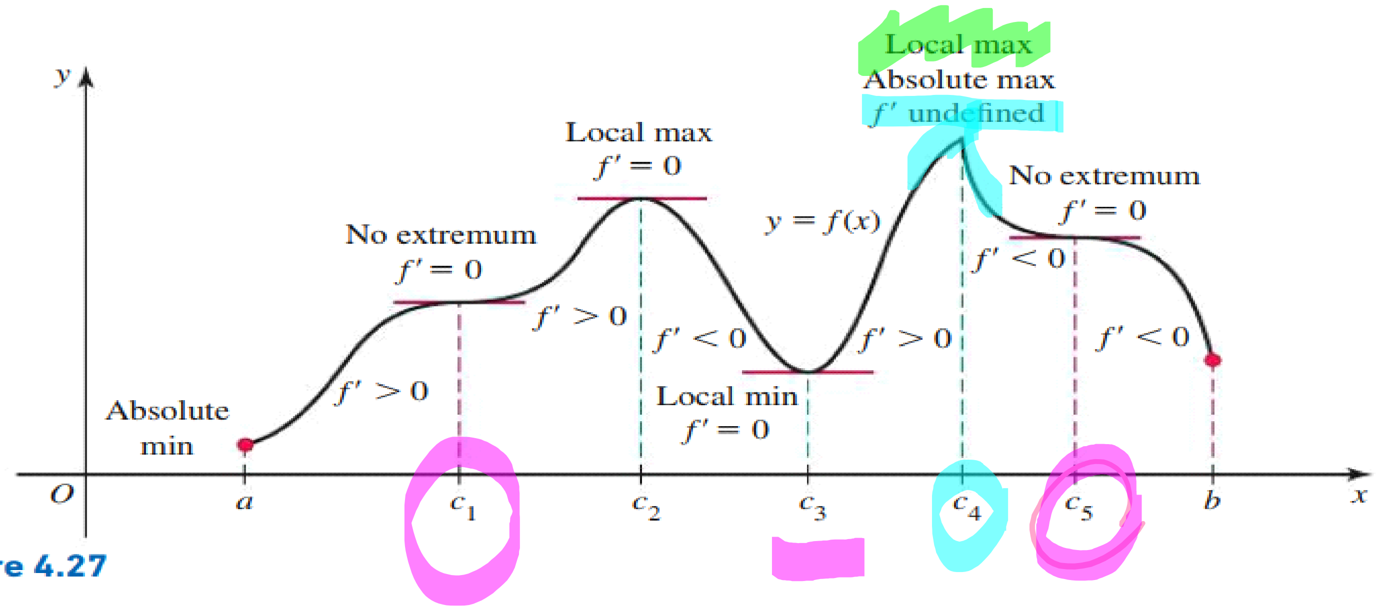

Figure 4.27

First Derivative Test The observations used to interpret Figure 4.27 are summarized in a powerful test for identifying local maxima and minima.

peak

THEOREM 4.8 First Derivative Test

Assume f is continuous on an interval that contains a critical point c, and assume f is differentiable on an interval containing c, except perhaps at c itself.

- If f' changes sign from positive to negative as x increases through c, then f has a **local maximum** at c.
- If f' changes sign from negative to positive as x increases through c, then f has a **local minimum** at c.
- If f' is positive on both sides near c or negative on both sides near c, then f has no local extreme value at c.

valley

EXAMPLE 3 **Using the First Derivative Test** Consider the function

$$f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1.$$

a. Find the intervals on which f is increasing and those on which it is decreasing.

b. Identify the local extrema of f.

a. Find the derivative of $f(x) \to f'(x)$

$$f'(x) = 3\cdot 4\cdot x^3 - 4\cdot 3\cdot x^2 - 6\cdot 2\cdot x + 12 + 0$$

$$= \underbrace{12x^3 - 12x^2}\underbrace{-12x+12}$$

$$= \underbrace{12x^2(x-1)}\underbrace{-12(x-1)} = 12(x-1)[x^2-1]$$

$$= 12\underbrace{(x-1)(x-1)}(x+1) = 12(x-1)^2(x+1)$$

Find the critical P. $\underbrace{f'(x)=0}_{x=1,-1}$ or $\underbrace{\text{DNE}}_{\text{none}}$

f is poly.

$12, (x-1)^2$ are pos

$$f'(x) = 12(x-1)^2(x+1) = 0$$

$f'(-2)$

crit. p. are $x=1$, $x=-1$

Construct **sign chart** on $f'(x)$

	$-\infty$	test p. -2	-1	0	1	2	$+\infty$
sign of $f'(x)$		neg.		pos.		pos.	
behavior of $f(x)$		decreasing		increasing			

Local MIN at $x=-1$

(sign change of f' from ⊖ to ⊕)

$f'(-2) \to ⊖$; $f'(0) \to ⊕$; $f'(2) \to ⊕$

* $f(x)$ is decreasing on $(-\infty, -1)$
* $f(x)$ is increasing on $(-1, \infty)$
* horizontal tangent at $x = -1, 1$

EXAMPLE 4 **Extreme values** Find the local extrema of the function $g(x) = x^{2/3}(2 - x) = 2x^{2/3} - x^{5/3}$

$\frac{1}{\sqrt[3]{x}}$

Step1) $g'(x) = \left(2x^{2/3} - x^{5/3}\right)' = 2 \cdot \frac{2}{3} \cdot x^{-1/3} - \frac{5}{3} \cdot x^{2/3}$

$= \frac{4}{3\sqrt[3]{x}} - \frac{5x^{2/3}}{3} = \frac{4 - 5x}{3 \cdot \sqrt[3]{x}}$

$\left(\sqrt[3]{x} = x^{1/3}\right)$

Step2) Find critical P. are $x = 4/5$, $x = 0$

$g'(x) = 0$ or DNE

$g'(x) = \frac{4 - 5x}{3 \cdot \sqrt[3]{x}} = \frac{0}{1} \Rightarrow 4 - 5x = 0$

$\boxed{x = \frac{4}{5}}$

$g'(x) \to$ DNE : $3 \cdot \sqrt[3]{x} = 0 \to \boxed{x = 0}$

Step 3) Construct the sign chart, test P.

	$-\infty$	-2	0	$1/2$	$4/5$	1	$+\infty$
sign of $g'(x)$		$-$	0	$+$	0	$-$	
$g(x)$			local min		local max		

$$g'(-2) \to \frac{\oplus}{\ominus} \to \ominus$$

$$g'\left(\tfrac{1}{2}\right) \to \frac{\oplus}{\oplus} \to \oplus$$

$$g'(1) \to \frac{\ominus}{\oplus} \to \ominus$$

$g(x)$ is decr. on $(-\infty, 0), (4/5, \infty)$; $g(x)$ is incr. on $(0, 4/5)$

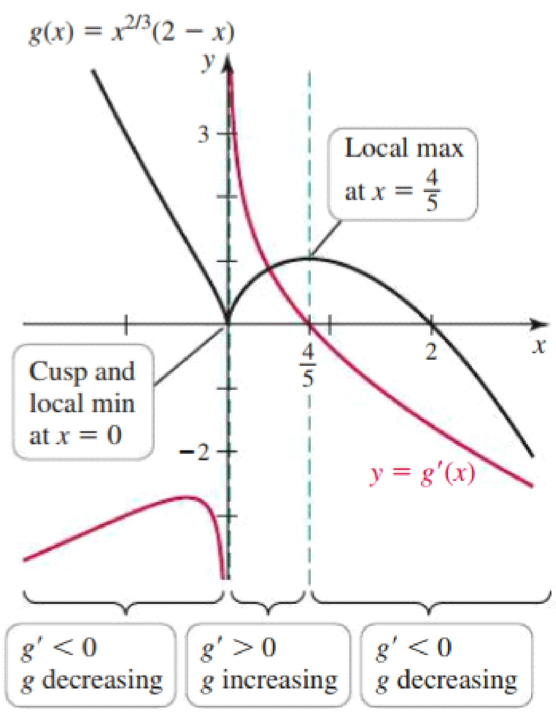

Figure 4.31

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EXAMPLE 5 **Finding an absolute extremum** Verify that $f(x) = x^x$ has an absolute extreme value on its domain.

$f(x) = x^x$

$y = ?$

$f'(x) = ?$

$\ln y = \ln x^x \Rightarrow y = e^{x \ln x}$

D. $(0, \infty)$

$\ln y = x \cdot \ln x$

$(\ln y)' = (x \cdot \ln x)'$

$\frac{y'}{y} = 1 \cdot \ln x + x \cdot \frac{1}{x}$

$\frac{y'}{y} = \ln x + 1 \Rightarrow y' = y \cdot (\ln x + 1)$

$y' = x^x \cdot (\ln x + 1)$

$f'(x) = x^x \cdot (\ln x + 1)$

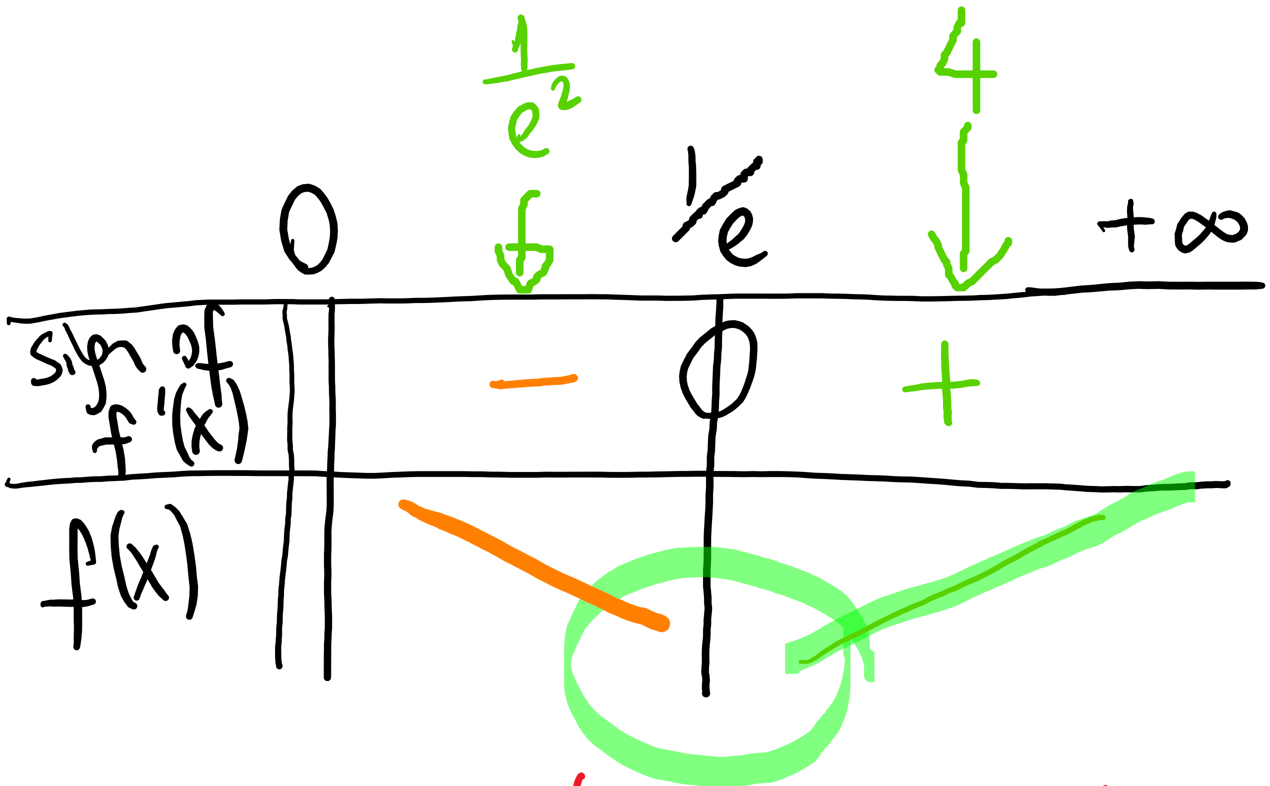
crit. P. $f'(x) = 0$ or $\overbrace{\text{DNE}}^{\text{none}}$

$\cancel{x^x = 0}$ or $\ln x + 1 = 0$

$\ln_e x = -1 \Rightarrow x = e^{-1} = \frac{1}{e}$

sign chart

test p.



Local/abs. min at $x = 1/e$

$$f'(x) = x^x \cdot (\ln x + 1)$$

test p.

$x = \frac{1}{e^2}, 4$

$f'(4) \Rightarrow +$

$f'\left(\frac{1}{e^2}\right) \Rightarrow \left(\frac{1}{e^2}\right)^{1/e^2} \cdot \underbrace{(\ln e^{-2} + 1)}_{-}$

THEOREM 4.9 One Local Extremum Implies Absolute Extremum

Suppose f is continuous on an interval I that contains exactly one local extremum at c.

- If a local maximum occurs at c, then $f(c)$ is the absolute maximum of f on I.
- If a local minimum occurs at c, then $f(c)$ is the absolute minimum of f on I.