Exp4) Angle of Elevation

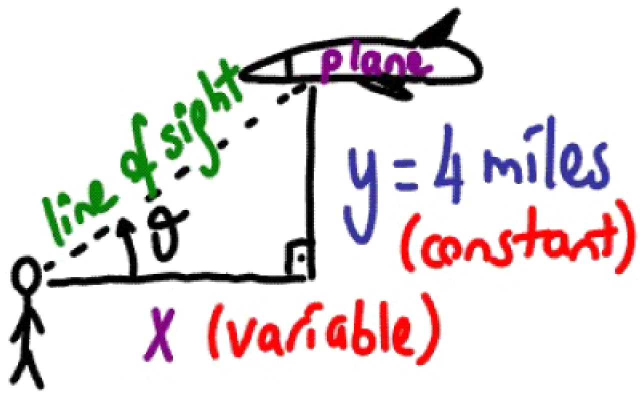
A plane is flying at a constant speed of 400 $\frac{\text{miles}}{\text{hr}}$ at a constant altitude of 4 miles. At what rate is the angle of elevation of my line of sight changing wrt time when the horizontal

specific case

distance between the approaching plane and my location is exactly 3 miles?

Solution

① Draw a figure & identify given/asked



x → horizontal distance between plane and the person

y → constant altitude

θ → angle of elevation

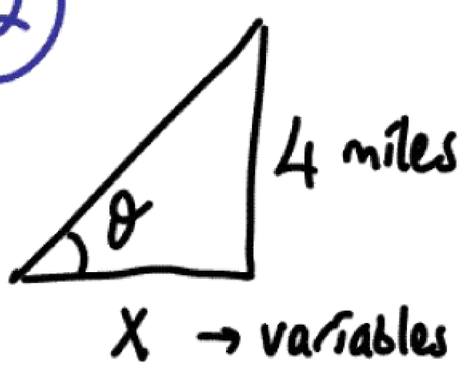
* height of the person is insignificant compared to the vertical distance y = 4 miles

Given: when x = 3 miles, $\frac{dx}{dt} = -400\ \frac{\text{miles}}{\text{hr.}}$

(negative since x is decreasing by time)

Asked: @ that point in time, how's θ changing? $\left(\frac{d\theta}{dt} = ?\right)$

②



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$$\tan\theta = \frac{\text{Opp}}{\text{Adj.}} = \frac{4}{x}$$

(3) $\tan\theta = \dfrac{4}{x} \Rightarrow \tan\theta = 4 \cdot x^{-1}$

differentiate both sides wrt time

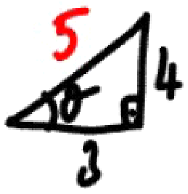
$$\sec^2\theta \cdot \frac{d\theta}{dt} = 4 \cdot (-1) \cdot x^{-2} \cdot \frac{dx}{dt}$$

(both x, θ changes wrt time)

(4) when $x = 3$ miles, $\dfrac{dx}{dt} = -400 \dfrac{\text{miles}}{\text{hr.}}$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = -4 \cdot x^{-2} \cdot \frac{dx}{dt}$$

Find $\sec\theta$ first.



$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{5}{3}$

$$\left(\frac{5}{3}\right)^2 \cdot \frac{d\theta}{dt} = -4 \cdot 3^{-2} \cdot (-400)$$

$$\frac{9}{25} \cdot \frac{25}{9} \cdot \frac{d\theta}{dt} = +4 \cdot \frac{1}{9} \cdot (+400) \cdot \frac{9}{25}$$

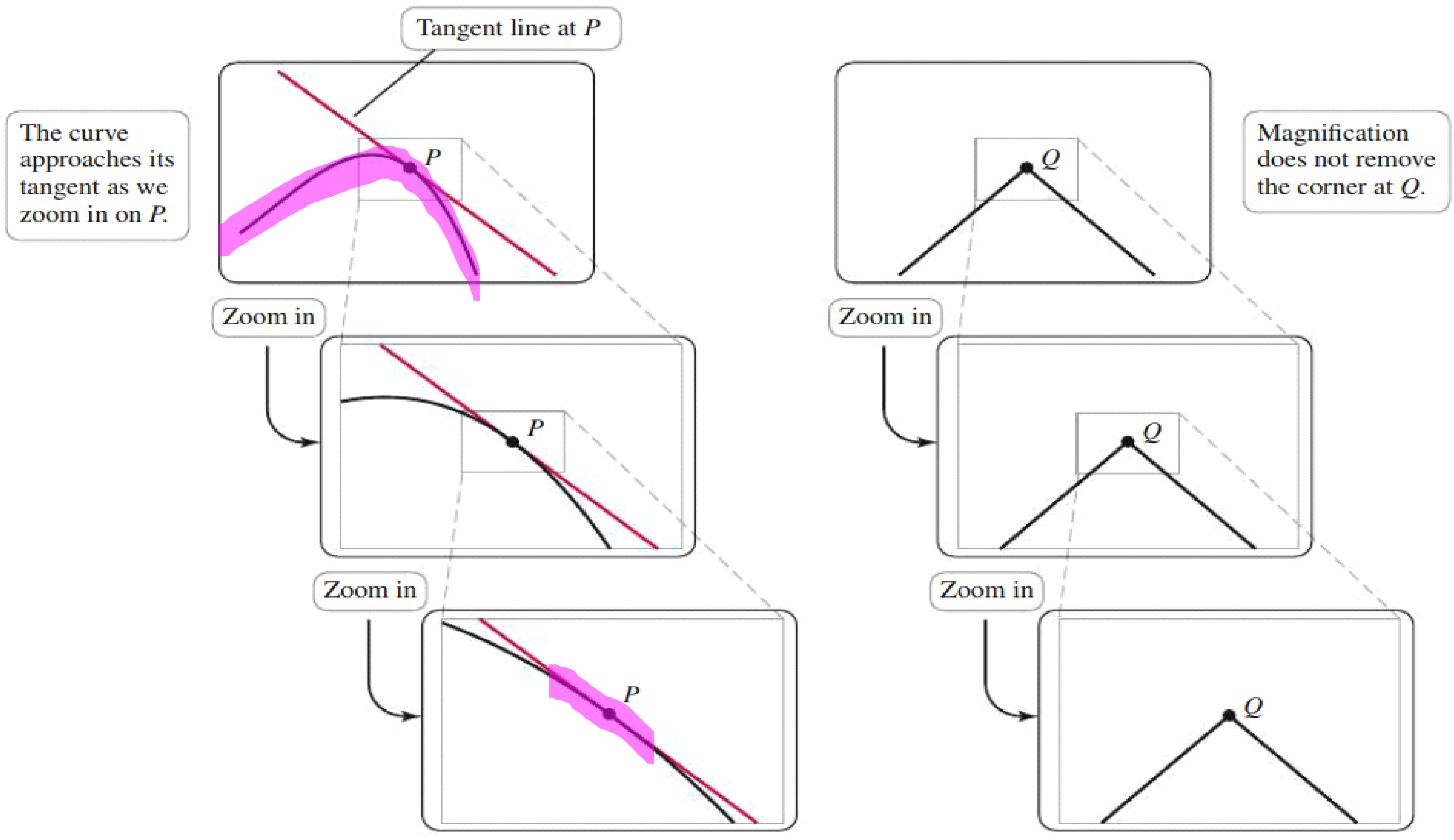
$$\frac{d\theta}{dt} = 4 \cdot \frac{1}{\cancel{9}} \cdot \overset{16}{\cancel{400}} \cdot \frac{\cancel{9}}{\cancel{25}}$$

$$\frac{d\theta}{dt} = 64 \ \frac{\text{rad}}{\text{hr}}$$ (in Trig, angles are in radians)

The angle of elevation is increasing at a rate of 64 rad/hr.

4.6 Linear Approximation and Differentials

Function is diff. at P **Function is NOT diff. at Q (corner)**



Imagine plotting a smooth curve with a graphing utility. Now pick a point P on the curve, draw the line tangent to the curve at P, and zoom in on it several times. As you successively enlarge the curve near P, it looks more and more like the tangent line (Figure 4.67a). This fundamental observation—that smooth curves appear straighter on smaller scales—is called *local linearity*; it is the basis of many important mathematical ideas, one of which is *linear approximation*.

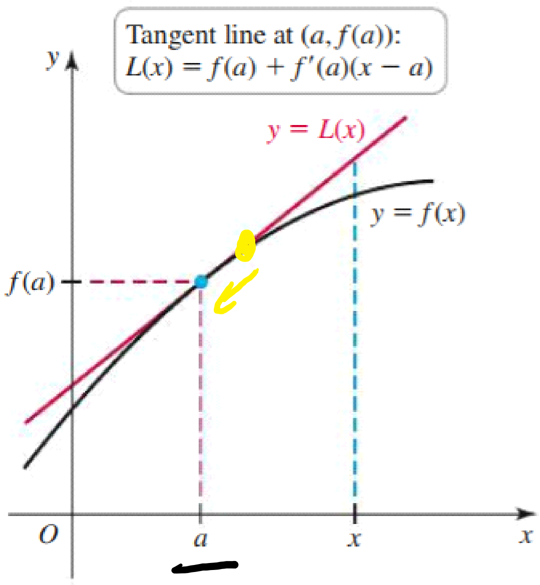


Figure 4.68

Linear Approximation

Figure 4.67a suggests that when we zoom in on the graph of a smooth function at a point P, the curve approaches its tangent line at P. This fact is the key to understanding linear approximation. The idea is to use the line tangent to the curve at P to approximate the value of the function at points near P. Here's how it works.

f is diff. on I containing a

The slope of the line tangent to the curve at $(a, f(a))$ is $\underline{f'(a)}$

Eq. of the tangent line is:

$$y - f(a) = f'(a)(x-a) \Rightarrow y = \underbrace{f(a) + f'(a)(x-a)}_{L(x)}$$

$$y - y_1 = m_{tan}\big|_{x=a}(x - x_1)$$

This approximation improves as x approaches a.

DEFINITION Linear Approximation to f at a

Suppose f is differentiable on an interval I containing the point a. The **linear approximation** to f at a is the linear function

$$L(x) = f(a) + f'(a)(x - a), \quad \text{for } x \text{ in } I.$$

$L(x) = f(a) + f'(a)(x-a)$

y

EXAMPLE 2 **Linear approximations and errors**

ref. point

a. Find the linear approximation to $f(x) = \sqrt{x}$ at $x = 1$ and use it to approximate $\sqrt{1.1}$.

b. Use linear approximation to estimate the value of $\sqrt{0.1}$.

ref. point is NOT given!

$f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \frac{1}{2} \cdot x^{-1/2}$

$a = 1$; $f(1) = \sqrt{1} = 1$

$f'(1) = \frac{1}{2} \cdot 1^{-1/2} = \frac{1}{2}$

$L(x) = f(1) + f'(1)(x-1)$

$= 1 + \frac{1}{2}(x-1) = 1 + \frac{1}{2}x - \frac{1}{2} = \frac{1}{2}x + \frac{1}{2}$

$L(x) = \frac{1}{2}(x+1)$

$\sqrt{1.1} \approx L(1.1) = \frac{1}{2}(1.1+1) = 1.05$

linear approx.

calc.

Table 4.4

x	$L(x)$	Exact $\sqrt{x}$	Error
1.2	1.1	1.0954 . . .	4.6×10^{-3}
1.1	1.05	1.0488 . . .	1.2×10^{-3}
1.01	1.005	1.0049 . . .	1.2×10^{-5}
1.001	1.0005	1.0005 . . .	1.2×10^{-7}

How does a calculator evaluates sqrt(1.2) etc?

The calculator uses an approximation! In fact, calculators and computers use approximations all the time to evaluate mathematical expressions; they just use higher-degree approximations.

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$L(x) = \frac{1}{2}(x+1)$

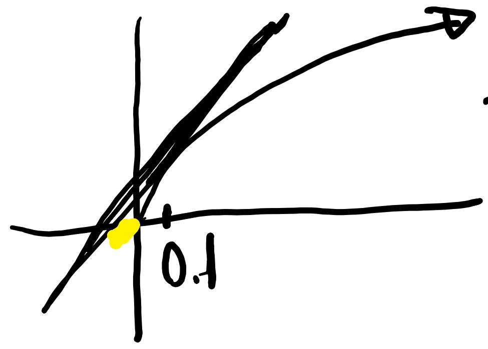
$\to x = 1$

b) $\sqrt{0.1} \approx L(0.1) = \frac{1}{2}(0.1+1) = 0.55$

Calculator:

$\sqrt{0.1} \approx 0.3162\ldots$ way off!

$\sqrt{0}$



$f(x) = \sqrt{x}\ ;\ x \geq 0$

$f'(0)$ is undef.

$f(x)$ is NOT diff. at $x=0$

$\sqrt{0.10} \to$

* find another ref. x-value closer to 0.10 AND easy to compute

$\sqrt{0.1} \approx ?$

ref.# $\to$ 0.09

$\sqrt{0.1} \approx L(0.1) = f(a) + f'(a)\cdot(x-a)$ $\leftarrow$ ref.#

$= f(0.09) + f'(0.09)(x-0.09)$

$= 0.3 + \frac{10}{6}(x - 0.09)$

$f(x) = \sqrt{x},\ f'(x) = \frac{1}{2}\cdot x^{-1/2}$

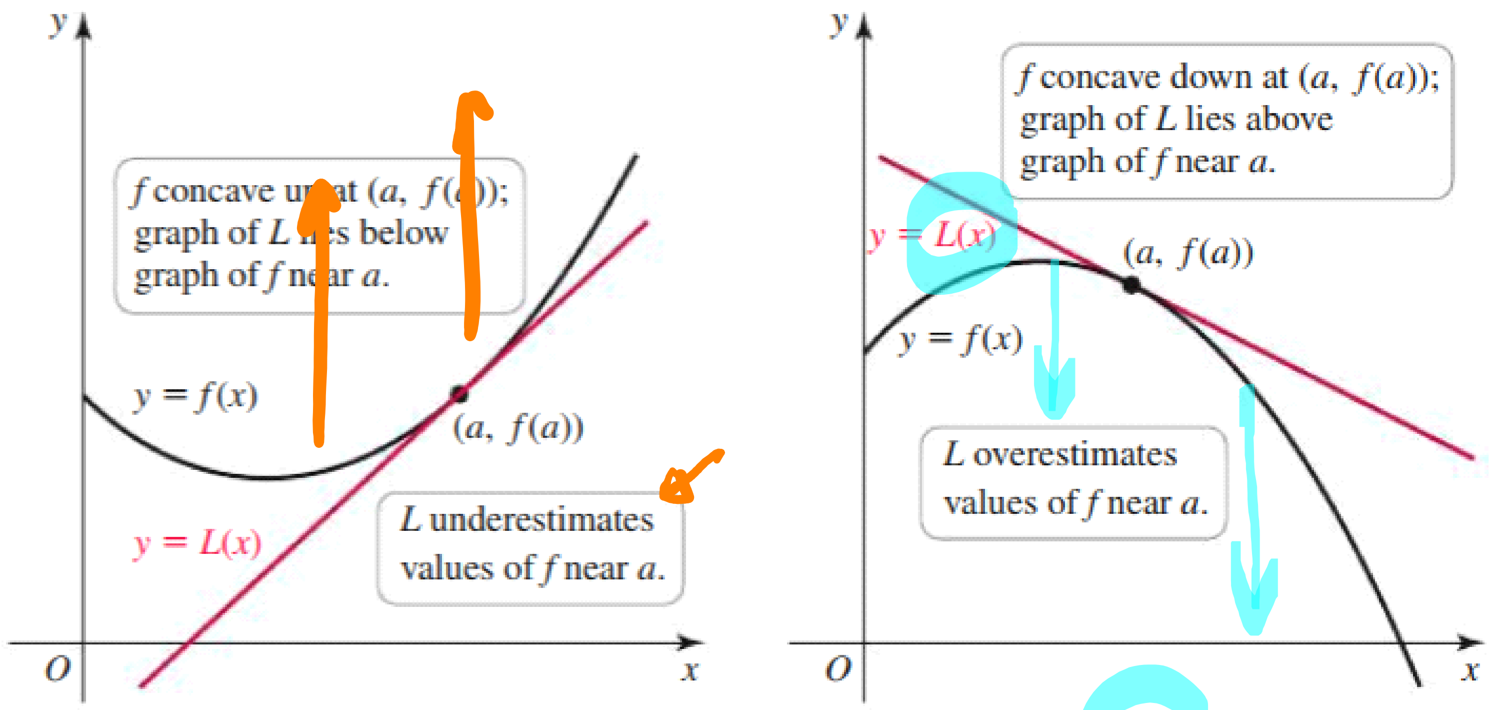
$f(0.09) = \sqrt{0.09} = 0.3$

$f'(0.09) = \frac{1}{2\sqrt{0.09}} = \frac{1}{2\times 0.3} = \frac{1}{0.6} = \frac{1}{\frac{6}{10}} = \frac{10}{6}$

$L(0.1) = 0.3 + \frac{10}{6}(0.1 - 0.09) = \frac{19}{60} \approx 0.3167$

$$L(0.1) = 0.3 + \frac{10}{6}(0.1 - 0.09) = \frac{19}{60} \approx 0.3167$$

Linear Approximation and Concavity



(a) (b)

Concave up: $f''(x) > 0$

L underestimates

Concave down: $f''(x) < 0$

L overestimates

EXAMPLE 4 **Linear approximation and concavity**

a. Find the linear approximation to $f(x) = x^{1/3}$ at $x = 1$ and $x = 27$.

b. Use the linear approximations of part (a) to approximate $\sqrt[3]{2}$ and $\sqrt[3]{26}$.

c. Are the approximations in part (b) overestimates or underestimates?

d. Compute the error in each approximation of part (b). Which error is greater? Explain.

$$L(x) = f(a) + f'(a)(x-a)$$

a. $f(1) = 1$

$a = 1$ ref. p.

$f(x) = x^{1/3}$

$f'(x) = \frac{1}{3} \cdot x^{-2/3}$

$f'(1) = \frac{1}{3}$

$$L(x) = f(1) + f'(1)(x-1)$$
$$= \underbrace{1 + \frac{1}{3}(x-1)}$$

b. $L(2) = 1 + \frac{1}{3}(2-1) = \frac{4}{3} \approx 1.\bar{3}$

c. $f''(x) = ?$

$f''(x) = \frac{1}{3} \cdot \frac{-2}{3} \cdot x^{-5/3} = \frac{-2}{9} x^{-5/3}$ for $x > 0$

$f''(x) < 0$ for $x > 0$

↳ concave down

L overestimates the values of f at a.

d. $\left| L(2) - \sqrt[3]{2} \right| \approx 0.073$