EXAMPLE 4 **Linear approximation and concavity**

a. Find the linear approximation to $f(x) = x^{1/3}$ at $x = 1$ and $x = 27$.
b. Use the linear approximations of part (a) to approximate $\sqrt[3]{2}$ and $\sqrt[3]{26}$.
c. Are the approximations in part (b) overestimates or underestimates?
d. Compute the error in each approximation of part (b). Which error is greater? Explain.

$$y - y_1 = m(x - x_1)$$

$$y = m\underbrace{(x - x_1)}_{(x-a)} + y_1$$

$$L(x) = f(a) + f'(a)(x-a)$$

a. $f(1) = 1$

$f(x) = x^{1/3}$

$f'(x) = \frac{1}{3} \cdot x^{-2/3}$

$f'(1) = \frac{1}{3}$

$a = 1$ ref. p.

$$L(x) = f(1) + f'(1)(x-1) = 1 + \frac{1}{3}(x-1)$$

b. $\sqrt[3]{2} \approx L(2) = 1 + \frac{1}{3}(2-1) = \frac{4}{3} \approx 1.\bar{3}$

c. $f''(x) = ?$

$f''(x) = \frac{1}{3} \cdot \frac{-2}{3} \cdot x^{-5/3} = \frac{-2}{9} x^{-5/3}$ for $x > 0$

$f''(x) < 0$ for $x > 0$

↳ concave down

L overestimates the values of f at a.

d. $\left| L(2) - \sqrt[3]{2} \right| \approx 0.073$

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a. $L(x) = f(a) + f'(a)(x-a)$

$f(x) = x^{1/3}$

$x = 27$

$f(27) = (3^3)^{1/3} = 3$

$f'(x) = \frac{1}{3} x^{-2/3}$

$f'(27) = \frac{1}{3} \cdot (3^3)^{-2/3} = \frac{1}{3} \cdot 3^{-2} = \frac{1}{3} \cdot \frac{1}{3^2} = \frac{1}{27}$

The linear approx. at $x=27$ is:

$$\boxed{L(x) = 3 + \frac{1}{27}(x-27)}$$

b. $\sqrt[3]{26} \simeq ?$

$\sqrt[3]{26} \approx L(26) = 3 + \frac{1}{27}(26-27) = 3 - \frac{1}{27}$

$= \frac{3}{1} - \frac{1}{27} = \frac{81-1}{27} = \frac{80}{27} \approx 2.963$

(27)

error

d. $\left| L(26) - \sqrt[3]{26} \right| = \left| 2.963 - 2.9625 \right|$

$L(26)$ ↓ approx.

$\sqrt[3]{26}$ ↓ exact w/ calc.

≈ 0.0005

0.073 error in approx. $\sqrt[3]{2}$ $>$ 0.0005 error in approx. $\sqrt[3]{26}$

ALTERNATIVE EXPLANATION

curvature $f''(x)$ (large curvature causes more rapid change)

$$f''(x) = \frac{-2}{9} \cdot x^{-5/3}$$

$|f''(1)| \approx 0.22$

$|f''(27)| \approx 0.00091$

ref. p

d) Explanation

The curvature of f is greater at x= 1 than at x=27.

This says that the error of approximating 2 (by using the reference point of x=1) is greater than the error of approximating 26 (by using the reference point of x=27).

This also says that the approximation of f(26) is more accurate than the approximation of f(2).