Ch 4.9 Antiderivatives

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4.9 Antiderivatives

The goal of differentiation is to find the derivative f' of a given function f. The reverse process, called *antidifferentiation*, is equally important: Given a function f, we look for an *antiderivative* function F whose derivative is f; that is, a function F such that $F' = f$.

> **DEFINITION Antiderivative**
>
> A function F is an **antiderivative** of f on an interval I provided $F'(x) = f(x)$, for all x in I.

$f(x) \to f'(x)$

$f(x) = ?$

$f(x) \to$ original f.

$F(x) \to$ antiderivative f.

Inverse (Backward) Thinking

$f(x) = 2x$ the antiderivative of $f(x)$ is $F(x) = x^2$?

$F'(x) = \frac{d}{dx}(x^2) = (x^2)' = 2x \hookrightarrow f(x)$

$f(x) = 2x$, another antider. of $f(x)$ is $F(x) = x^2 + \underbrace{10}$

$(x^2+10)' = 2x$

$F(x) = x^2 - 10^3$

$F(x) = x^2 + \pi^4$

> **THEOREM 4.15 The Family of Antiderivatives**
>
> Let F be any antiderivative of f on an interval I. Then *all* the antiderivatives of f on I have the form $F + C$, where C is an arbitrary constant.

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Inverse trig. functions are NOT in the curriculum

EXAMPLE 1 **Finding antiderivatives** Use what you know about derivatives to find all antiderivatives of the following functions.

a. $f(x) = 3x^2$ **b.** ~~$f(x) = \frac{1}{1 + x^2}$~~ **c.** $f(t) = \sin t$

a. $f(x) = 3x^2$ $F(x) = ?$

$F(x) = x^3 + C \rightarrow \frac{dF}{dx} = (x^3)' = 3x^2 = f(x)$ ✓

C → arbitrary constant

c. $f(t) = \sin t$ $F(t) = ?$

$\frac{dF}{dt} = f(t) = (\quad)'$

$(\cos t)' = -\sin t$

$(-\cos t)' = +\sin t$

$F(t) = -\cos t + C$

verify: $(-\cos t + C)' = \sin t + 0 \rightarrow f(t)$

Terminology:

Indefinite Integral

$$\int f(x) \cdot dx = F(x) + C$$

$\int$ integral sign

$f(x) \rightarrow$ integrand

$dx \rightarrow$ differential (x is the independent var.)

($x \rightarrow$ variable of integration)

$C \rightarrow$ constant of integration

$F(x) \rightarrow$ complete family of antiderivatives

Indefinite Integrals

The notation $\frac{d}{dx}(f(x))$ means *take the derivative of* $f(x)$ with respect to x. We need analogous notation for antiderivatives. For historical reasons that become apparent in the next chapter, the notation that means *find the antiderivatives of* f is the **indefinite integral** $\int f(x)\,dx$. Every time an indefinite integral sign $\int$ appears, it is followed by a function called the **integrand**, which in turn is followed by the differential dx. For now, dx simply means that x is the independent variable, or the **variable of integration**. The notation $\int f(x)\,dx$ represents *all* the antiderivatives of f. When the integrand is a function of a variable different from x—say, $g(t)$—we write $\int g(t)\,dt$ to represent the antiderivatives of g.

Using this new notation, the three results of Example 1 are written

$$\int 3x^2\,dx = x^3 + C, \quad \cancel{\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C}, \quad \text{and} \quad \int \sin t\,dt = -\cos t + C,$$

where C is an arbitrary constant called a **constant of integration**. The derivative formulas presented earlier in the text may be written in terms of indefinite integrals. We begin with the Power Rule.

$p=3$

$$\int x^3 \cdot dx = \frac{x^4}{4} + C$$

THEOREM 4.16 Power Rule for Indefinite Integrals

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + C,$$

where $p \neq -1$ is a real number and C is an arbitrary constant.

THEOREM 4.17 Constant Multiple and Sum Rules

Constant Multiple Rule: $\int cf(x)\,dx = c\int f(x)\,dx$, for real numbers c

Sum Rule: $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

The following example shows how Theorems 4.16 and 4.17 are used.

EXAMPLE 2 **Indefinite integrals** Determine the following indefinite integrals.

a. $\int \underbrace{(3x^5 + 2 - 5\sqrt{x})}_{f(x)}\,dx$ **b.** $\int \left(\frac{4x^{19} - 5x^{-8}}{x^2}\right) dx$ **c.** $\int (z^2+1)(2z-5)\,dz$

a. $$\int (3x^5 + 2 - 5\cdot x^{1/2})\,dx = \int 3x^5\cdot dx + \int 2\cdot dx - \int 5x^{1/2}\,dx$$

$$= 3\cdot\frac{x^6}{6} + 2x - 5\cdot\frac{x^{3/2}}{3/2} + C$$

$(2x)' = 2$

$$F(x) = \frac{x^6}{2} + 2x - \frac{10}{3}x^{3/2} + C$$

Table 4.9 Indefinite Integrals of Trigonometric Functions

1. $\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \int \cos x \, dx = \sin x + C$
2. $\frac{d}{dx}(\cos x) = -\sin x \quad \Rightarrow \quad \int \sin x \, dx = -\cos x + C$
3. $\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x \, dx = \tan x + C$
4. $\frac{d}{dx}(\cot x) = -\csc^2 x \quad \Rightarrow \quad \int \csc^2 x \, dx = -\cot x + C$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x \quad \Rightarrow \quad \int \sec x \tan x \, dx = \sec x + C$
6. $\frac{d}{dx}(\csc x) = -\csc x \cot x \quad \Rightarrow \quad \int \csc x \cot x \, dx = -\csc x + C$

EXAMPLE 3 **Indefinite integrals of trigonometric functions** Evaluate the following indefinite integrals.

a. $\int \sec^2 x \, dx$ **b.** $\int (2x + 3\cos x) \, dx$ **c.** $\int \frac{\sin x}{\cos^2 x} dx$

a. $\int \sec^2 x \cdot dx = \tan x + C$

b. $\int (2x + 3 \cdot \cos x) \, dx = x^2 + 3 \cdot \sin x + C$

c. $\int \frac{\sin x}{\cos^2 x} \cdot dx = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \cdot dx$

we used trig. to re-write it.

$= \int \tan x \cdot \sec x \cdot dx$

$= \sec x + C$

EXAMPLE 4 **Indefinite integrals involving trigonometric functions** Determine the following indefinite integrals.

a. $\int \left(\frac{2}{\pi}\sin x - 2\csc^2 x\right) dx$ **b.** $\int \frac{4\cos x + \sin^2 x}{\sin^2 x} dx$

a. π is constant # (3.14 ~)

$$\int \left(\frac{2}{\pi} \cdot \sin x - 2 \cdot \csc^2 x\right) \cdot dx$$

$$= \frac{2}{\pi} \cdot (-\cos x) - 2(-\cot x) + C$$

$$= -\frac{2}{\pi} \cdot \cos x + 2 \cdot \cot x + C$$

b. $\int \frac{4 \cdot \cos x + \sin^2 x}{\sin^2 x} \cdot dx = \int \left(\frac{4\cos x}{\sin^2 x} + 1\right) \cdot dx$

$$= 4 \int \frac{\cos x}{\underbrace{\sin x \cdot \sin x}} \cdot dx + \int 1 \cdot dx$$

use power rule

$$= 4 \int \cot x \cdot \csc x \cdot dx + \int x^0 \cdot dx$$

$$= -4 \cdot \csc x + \frac{x^1}{1} + C = -4\csc x + x + C$$

$$\int dx = \int 1 \cdot dx = x + C$$

$$(\ln(x))' = \frac{x'}{x} = \frac{1}{x}$$

Table 4.10 Other Indefinite Integrals

7. $\frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x\,dx = e^x + C$

8. $\frac{d}{dx}(\ln|x|) = \frac{1}{x} \Rightarrow \underbrace{\int \frac{dx}{x} = \ln|x| + C}$

$$\ln|-2| = \ln(2)$$

EXAMPLE 5 Additional indefinite integrals Evaluate the following indefinite integrals.

a. $\int \frac{dx}{x}$ **b.** $\int \frac{e^x}{3}\,dx$

$$\int \frac{1}{x}\cdot dx = \ln|x|$$

domain of $\ln(x)$ is $(0, \infty)$

a. $\int \frac{dx}{x} = \ln|x| + C$

$$(e^x)' = e^x$$

b. $\int \frac{e^x}{3}\cdot dx = \frac{1}{3}\cdot\int e^x \cdot dx = \frac{1}{3}\cdot e^x + C$

EXAMPLE 6 Indefinite integrals Determine the following indefinite integrals using Table 4.10.

a. $\int \frac{4x^3 + 2x}{3x^2}\,dx$

a. $\int \frac{4x^3}{3x^2}\cdot dx + \int \frac{2x}{3x^2}\cdot dx$

$= \int \frac{4}{3}\cdot x^1 \cdot dx + \int \frac{2}{3x}\cdot dx$

$= \boxed{\frac{2}{3}x^2 + \frac{2}{3}\cdot\ln|x| + C}$

$[\ln(x)]' = \frac{1}{x}$