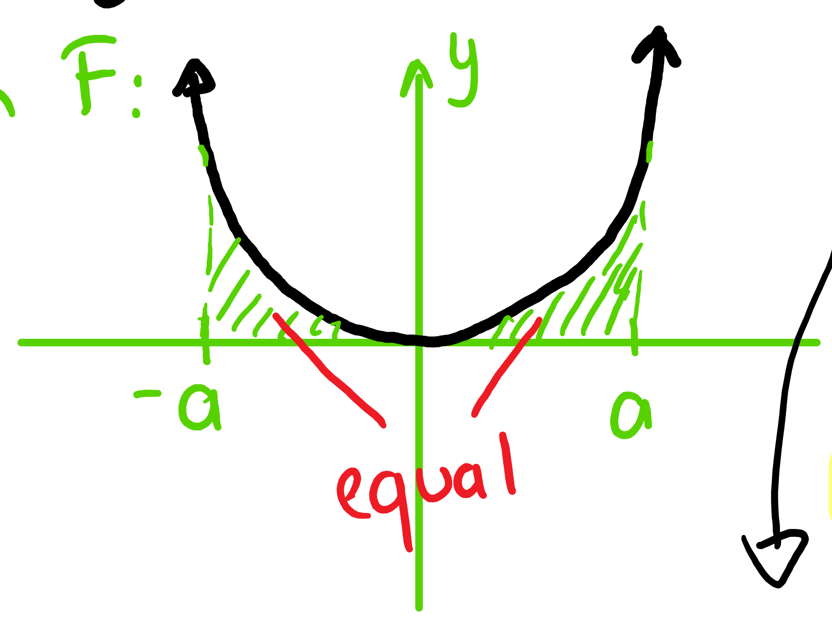
5.4. Working With Integrals

We will use the symmetry of functions to simplify integral calculations.

Even F:



$f(x) = x^2$

Symmetric wrt y-axis

$\boxed{f(x) = f(-x) \quad \text{Even F. Test}}$

$f(-x) = (-x)^2 = x^2 \qquad f(x) = f(-x)$ ✓

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = 2 \cdot \int_{0}^{a} f(x)\,dx$$

Odd F.

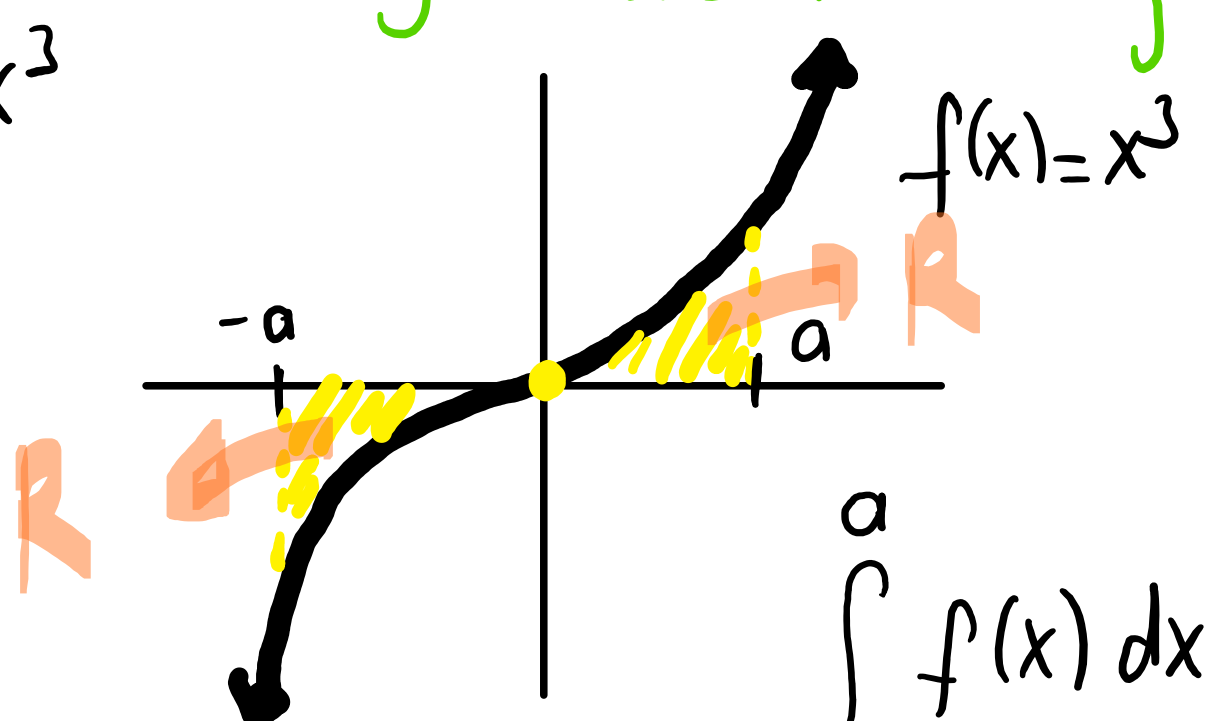
$\boxed{f(-x) = -f(x) \quad \text{Odd F. Test}}$

Sym. due to the origin

e.g: $f(x) = x^3$

$f(-x) = (-x)^3 = -x^3$

$-f(x) = -x^3$



$$\int_{-a}^{a} f(x)\,dx = 0$$

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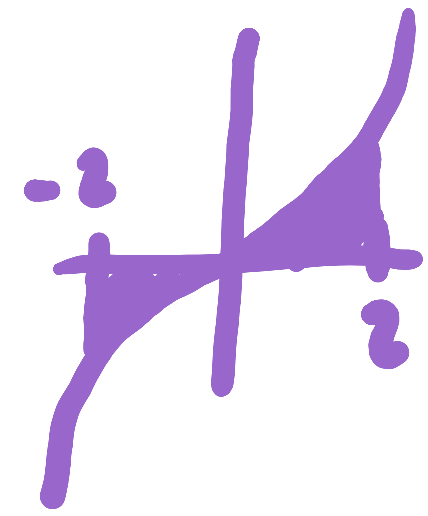
$\boxed{\text{E.g:}}$ Evaluate $\displaystyle\int_{-2}^{2} \underbrace{(x^4 - 3x^3)}_{\text{integral}} \underbrace{dx}_{\hookrightarrow \text{differential}}$

the original f. was NOT even / odd

$$\underbrace{\int_{-2}^{2} \overset{\nearrow \text{even}}{x^4}\, dx}_{} - 3 \underbrace{\int_{-2}^{2} \overset{\nearrow \text{odd}}{x^3}\, dx}_{}$$

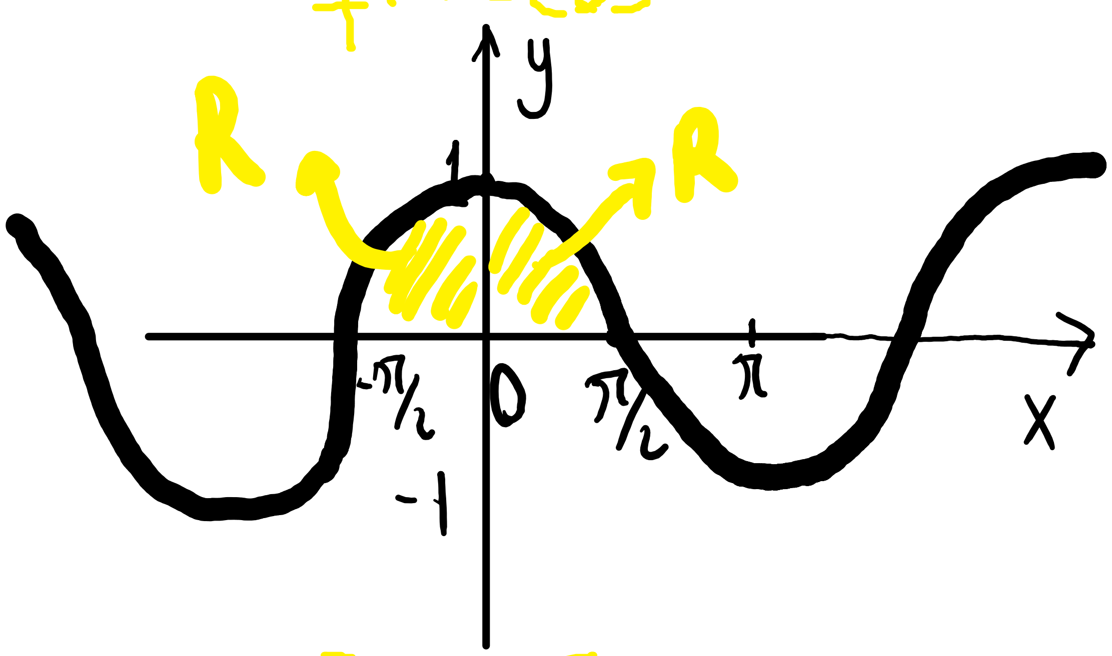


$$2 \cdot \int_0^2 x^4\, dx - \emptyset = 2 \cdot \frac{x^5}{5}\Big|_0^2 = 2\left(\frac{2^5}{5} - 0\right) = \frac{64}{5}$$

$\boxed{\text{E.g:}}$ Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} (\cos x - 4 \cdot \underline{\sin^3 x})\, dx$

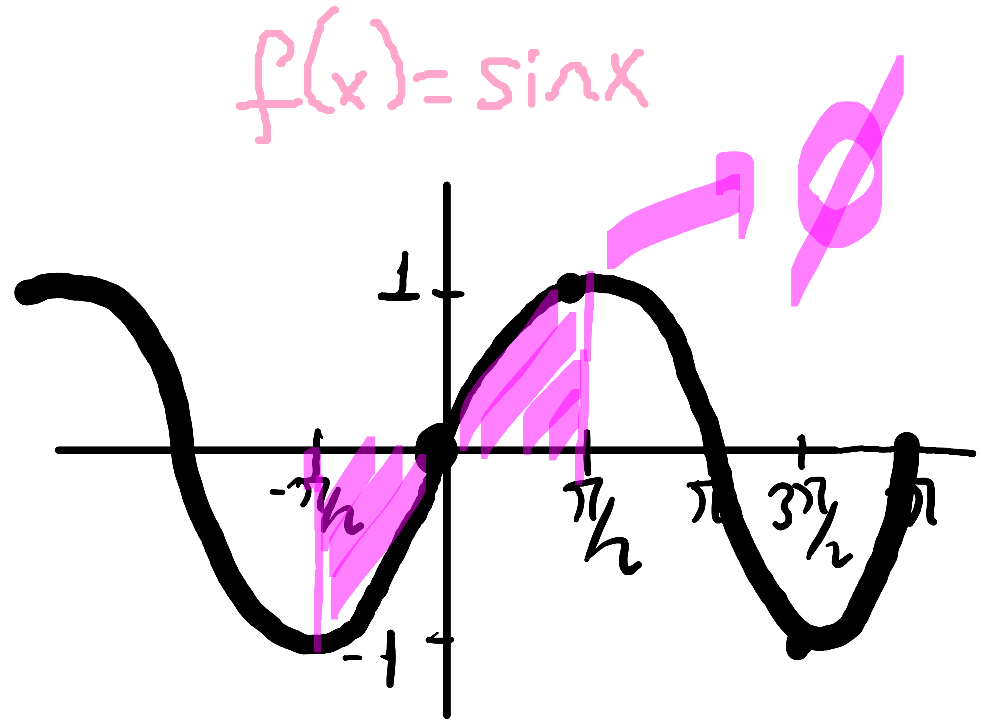
$\sin^3 x = (\sin x)^3$
$\neq \sin x^3$

$f(x) = \cos x$



Even F.

$f(x) = \sin x$



Odd F.

Odd F.

$$= \int_{-\pi/2}^{\pi/2} \cos x\,dx \;-\; 4\cdot \cancel{\int_{-\pi/2}^{\pi/2} \sin^3 x\,dx} = 2\cdot\underbrace{\int_0^{\pi/2}\cos x\,dx}_{\text{used sym.}}$$

$$2\int_0^{\pi/2}\cos x\,dx = 2\cdot \sin x\Big|_0^{\pi/2} = 2\left(\underbrace{\sin \pi/2}_{1} - \underbrace{\sin 0}_{0}\right) = 2$$

$\boxed{\text{You try it!}}$ Evaluate $\displaystyle\int_{-2}^{2}\left(\frac{x^3-4x}{x^2+1}\right)dx$

A) -2 B) 2 C) can not compute D) 0

Even Test: $f(x)=\underline{f(-x)}$ $\quad f(x)=\dfrac{x^3-4x}{x^2+1}$ $\quad f(-x)=\dfrac{(-x)^3-4(-x)}{(-x)^2+1}=\dfrac{-x^3+4x}{x^2+1}$

Odd Test: $-f(x)=\underline{f(-x)}$

$f(x)\neq f(-x)$ Not Even

$$-f(x) = -\left(\frac{x^3-4x}{x^2+1}\right) = \frac{-x^3+4x}{x^2+1}$$

$f(-x) = -f(x)$ Odd F.

$\emptyset$

Answer: D) 0

5.5. Substitution Rule

What are the antiderivatives of $\sin(x^2)$, $\frac{\sin x}{x}$, x^x?

Start with familiar derivative rules and work backward to find new derivative rules.

indef. integral

$$\int \cos 2x \cdot dx = \frac{\sin 2x}{2} + C \qquad \frac{d}{dx}\left(\frac{\sin(2x)}{2}\right) = \frac{\cos(2x)\cdot 2}{2}$$

Trial-and-error is not practical for complex integrals.

$\boxed{\text{E.g:}}$ Calculate $\int 4x^3 \cdot (x^4+5)^{10} dx$

$u = x^4+5$

$du = 4x^3 \cdot dx$

$$\int u^{10} \cdot du = \frac{u^{11}}{11} + C = \frac{(x^4+5)^{11}}{11} + C$$

Substitution Rule: Let $F' = f$

Then: $$\int f(u) \cdot u' \cdot dx = \int f(u) \cdot du = F(u) + C$$

$$u' = \frac{du}{dx}$$

You try it! Evaluate $\int x^9 \cdot \sin(x^{10}) \cdot dx$

A) $\dfrac{-\sin(x^{10})}{100} + C$ B) $\dfrac{\cos(x^{10})}{10} + C$ C) $\dfrac{-\cos(x^{10})}{11} + C$ D) $\dfrac{-\cos(x^{10})}{10} + C$

inside F.?

$$x^{10} = u$$

$$\frac{10 \cdot x^9 \cdot dx}{10} = \frac{du}{10} \Rightarrow x^9 \cdot dx = \frac{du}{10}$$

$$\int x^9 \cdot \sin(x^{10}) dx = \int \sin(x^{10}) \cdot x^9 \cdot dx = \int \sin(u) \cdot \frac{du}{10}$$

$$= \frac{-\cos u}{10} + C = \frac{-\cos(x^{10})}{10} + C$$

Answer: D

u-subs. for Definite Integrals

$$\int_a^b f(u)\cdot u'\cdot dx = \int_{u(a)}^{u(b)} f(u)\cdot du$$

Change
1) limits of integration
2) integrand
3) differential

$\boxed{\text{Exp:}}$ Evaluate $\int_0^{\pi/2} \sin^4 x\cdot\cos x\cdot dx$

$(\sin x)^4$, u

$u=\sin x$
$du=\cos x\cdot dx$

$x=\pi/2 \qquad u=\sin\pi/2=1$

$x=0 \qquad u=\sin 0=0$

$$\int_0^1 u^4\cdot du = \frac{u^5}{5}\Big|_0^1 = \frac{1^5}{5}-\cancel{\frac{0^5}{5}} = \frac{1}{5}$$

You try it! Evaluate $\int_0^4 \frac{x}{x^2+1}\,dx$

A) $\frac{\ln(17)}{2}$ B) $\ln(8.5)$ C) $\frac{\ln(19)}{2}$ D) $-\frac{\ln(17)}{4}$

$u = x^2+1$

$\frac{du}{2} = \frac{2x \cdot dx}{2} \Rightarrow \frac{du}{2} = x \cdot dx$

$x=4 \Rightarrow u = x^2+1 = 4^2+1 = 17$

$x=0 \Rightarrow u = x^2+1 = 1$

$$\boxed{(\ln(u))' = \frac{du}{u}}$$

$$\int_1^{17} \frac{du}{2u} = \frac{1}{2}\int_1^{17} \frac{du}{u}$$

$$= \frac{1}{2} \ln|u| \Big|_1^{17} = \frac{1}{2} \cdot \ln(u) \Big|_1^{17}$$

$$= \frac{1}{2}\left(\ln(17) - \underbrace{\ln(1)}_{\emptyset}\right) = \frac{1}{2} \cdot \ln(17)$$