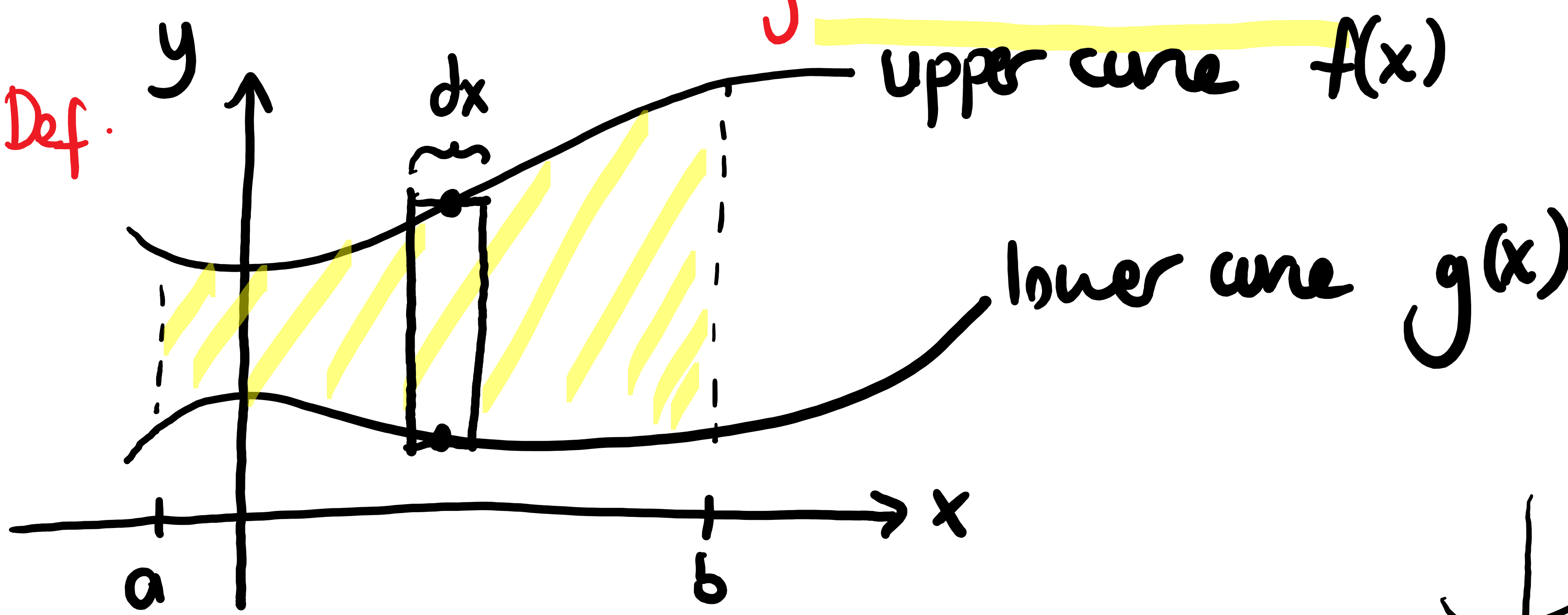
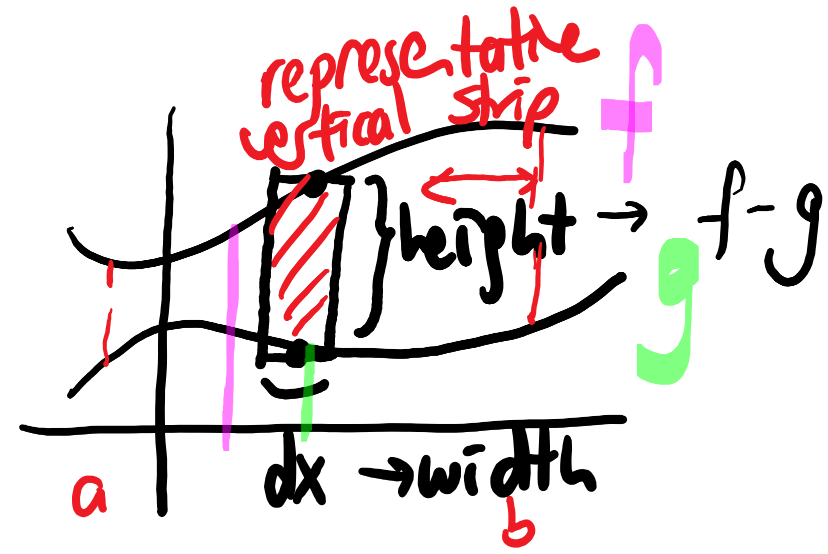
6.2. Regions Between Curves

Def.

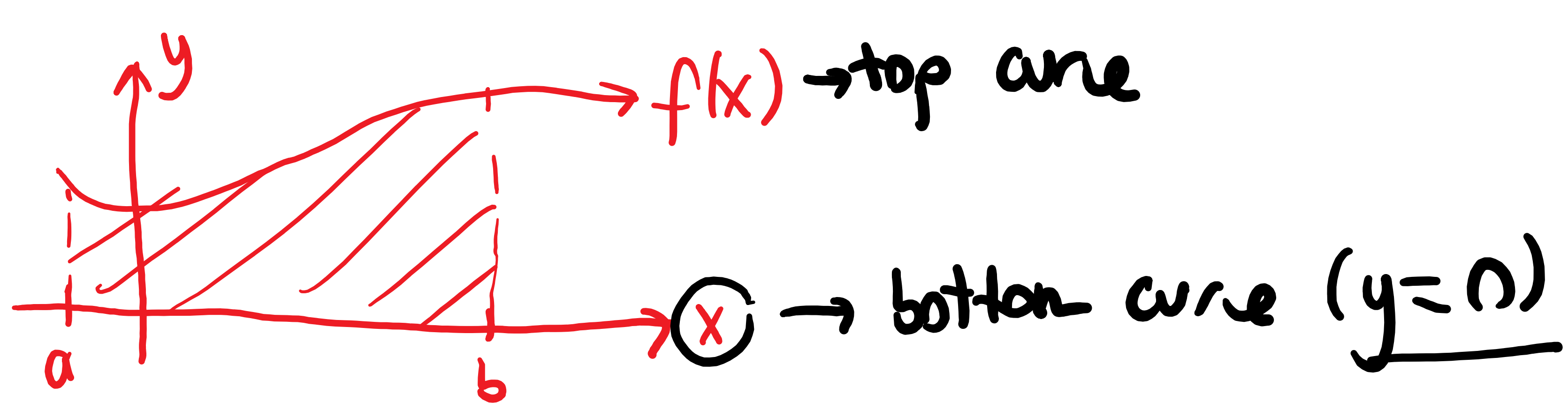


$$A = \int_a^b \big(\underbrace{f(x)}_{\text{top curve}} - \underbrace{g(x)}_{\text{bottom curve}}\big) \cdot dx$$



Let f, g be continuous f. with $f(x) \geq g(x)$ on $[a,b]$. The area of the region bounded by f & g on $[a,b]$ is A.

Recall:



$$\int_a^b (f(x) - 0) \cdot dx = \int_a^b f(x)dx$$

Steps:

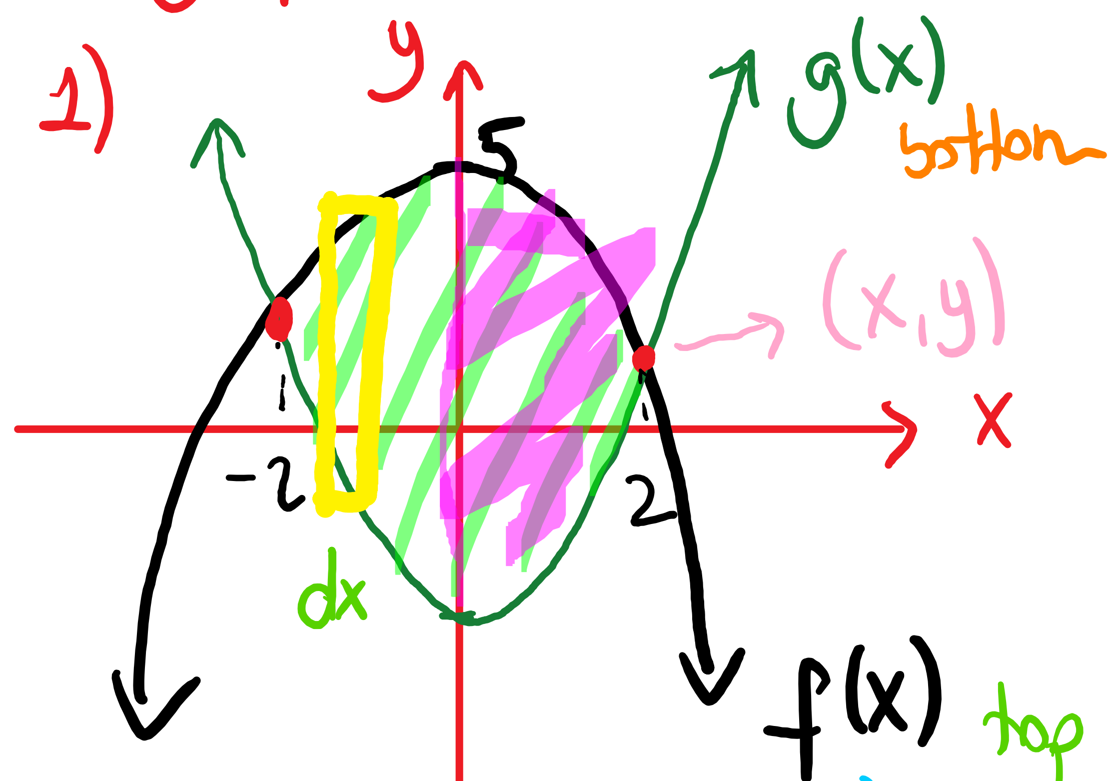
1) Sketch the area (need to know graphing parent functions!)
2) Determine the boundaries, a and b (find intersecting points of functions)
3) Identify top/bottom graphs per sub-interval
4) setup def. integral
5) Integrate

Friday, January 29, 2021 8:41 AM

Even F $f(x)=f(-x)$ $f(-x)=5-(-x)^2=5-x^2$

Expl) Find the area of the region bounded by the graphs of $f(x)=5-x^2$ and $g(x)=x^2-3$.

1)



both f, g are sym. wrt y-axis

(both f, g are even f.)
Even: $f(x)=f(-x)$

2) Boundaries?

$y=f(x)=g(x)$ to find intersect. P.

$5-x^2=x^2-3$

$\frac{8}{2}=\frac{2x^2}{2} \Rightarrow x^2=4 \Rightarrow x=\pm 2$

3) Top: $f(x)$, bottom: $g(x)$

4) $\int_{-2}^{2}(f-g)\,dx =$ 5) $\int_{-2}^{2}\left(5-x^2-(x^2-3)\right)dx = \int_{-2}^{2}(8-2x^2)\,dx$

$5-x^2-x^2+3$

$8-2x^2$

use sym. to re-write the integral

$$2\cdot\int_0^2(8-2x^2)\,dx = 2\cdot\left(8x-\frac{2x^3}{3}\right)\Bigg|_0^2 = 2\left(8\cdot 2-2\cdot\frac{2^3}{3}-0\right)$$

$$=2\left(\frac{16}{1_{(3)}}-\frac{16}{3}\right)=2\left(\frac{32}{3}\right)=\frac{64}{3}$$

Friday, January 29, 2021 8:52 AM

You try it!

Exp2) Find the area of the region bounded by the curve of $y = e^{2x} - 3\cdot e^{x} + 2$ and the $\underline{\text{x-axis}}$ $(y=0)$

A) $2 - \ln 2$ B) $1 + \ln 3$ C) $\frac{3}{2} - \ln 2$ D) $\frac{3}{2} - 2\cdot\ln 2$ E) $\frac{-3}{2} + \ln 2$

$$e^{2x} - 3\cdot e^{x} + 2 = 0$$

$$-2 \quad -1$$

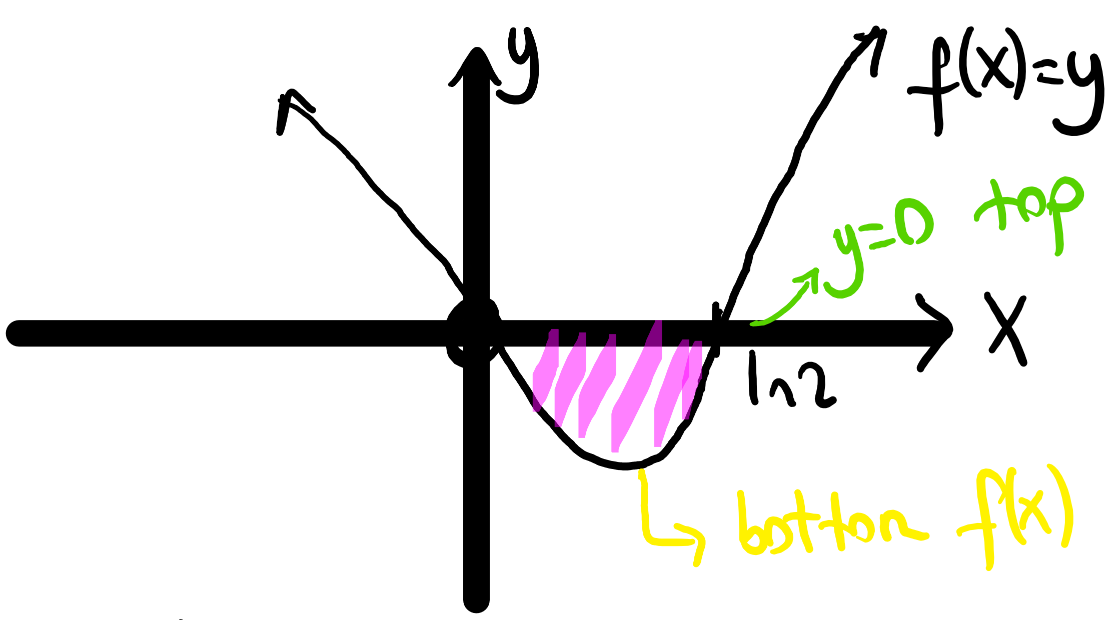
$$(e^{x} - 2)\cdot(e^{x} - 1) = 0$$

$$e^{x} = 2 \qquad\qquad e^{x} = 1$$

$$\ln e^{x} = \ln 2 \qquad\qquad \ln e^{x} = \ln 1$$

$$x \cdot \cancel{\ln e}^{\,1} = \ln 2 \qquad\qquad x\cdot \ln e = 0$$

zeros $\quad x = \ln 2 \qquad x = 0$ (boundaries / intersecting P.)



$$\int_0^{\ln 2} (\underbrace{0}_{\text{top}} - \underbrace{(e^{2x} - 3e^{x} + 2)}_{\text{bottom}})\cdot dx$$

$$= \int_0^{\ln 2} (-e^{2x} + 3\cdot e^{x} - 2)\,dx = \left(\frac{-e^{2x}}{2} + 3\cdot e^{x} - 2x\right)\Bigg|_0^{\ln 2}$$

Friday, January 29, 2021 9:04 AM

$$2$$

$$= \left(\frac{-e^{2\cdot\ln 2}}{2} + 3\cdot e^{\ln 2} - 2\cdot\ln 2 - \left(\frac{-e^0}{2} + 3\cdot e^0 - 0 \right) \right)$$

$$4$$

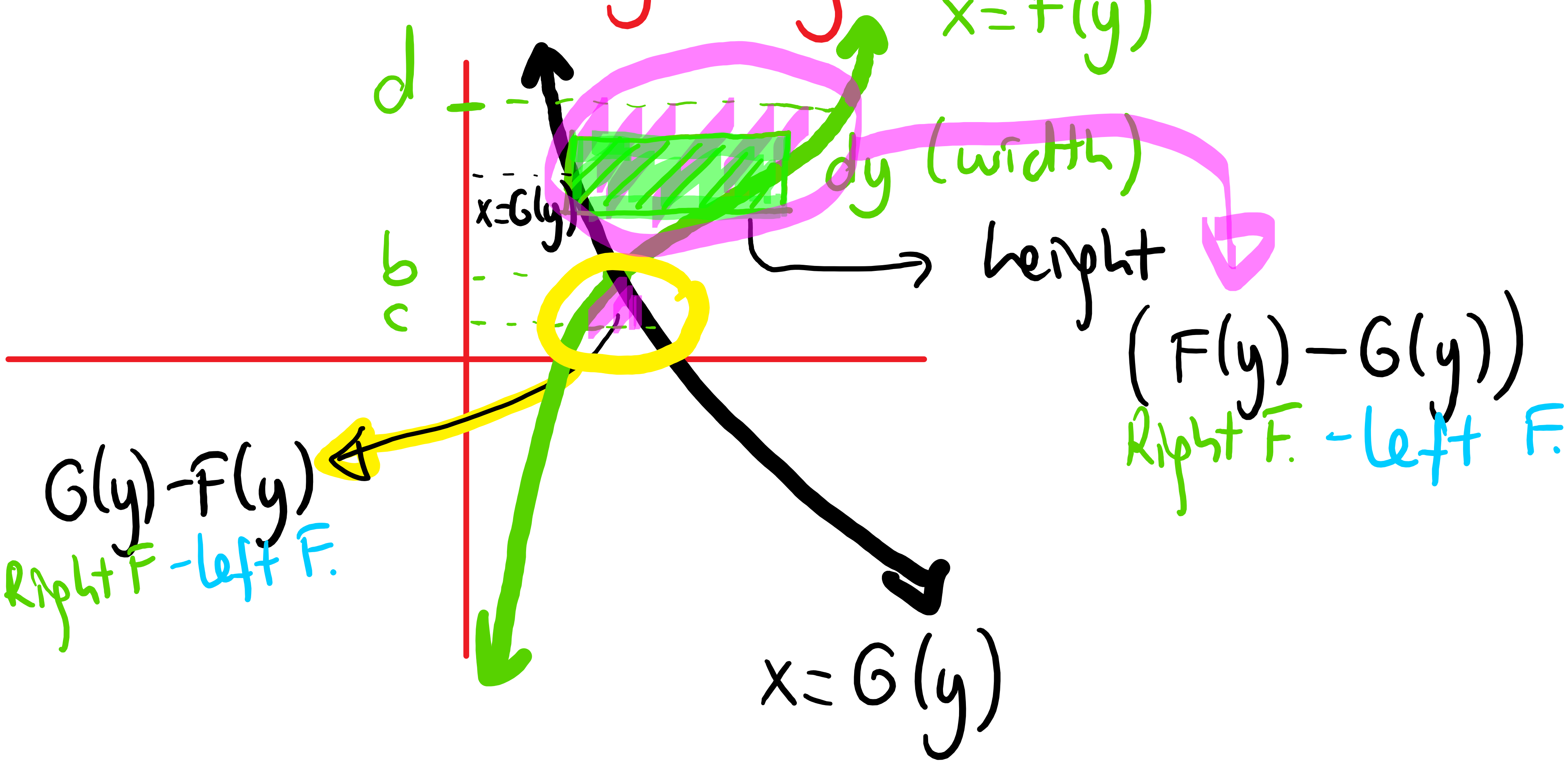
$$= \left(-\frac{e^{\ln 4}}{2} + 3\cdot 2 - 2\cdot\ln 2 - \left(-\frac{1}{2} + \frac{3}{1}_{(2)} \right) \right)$$

$$= \left(-\frac{4}{2} + 6 - 2\cdot\ln 2 - \frac{5}{2} \right) = \underbrace{-2+6}_{4} - 2\cdot\ln 2 - \frac{5}{2}$$

$$= 4 - \frac{5}{2} - 2\ln 2 \qquad = \frac{3}{2} - 2\cdot\ln 2$$

Friday, January 29, 2021 9:09 AM

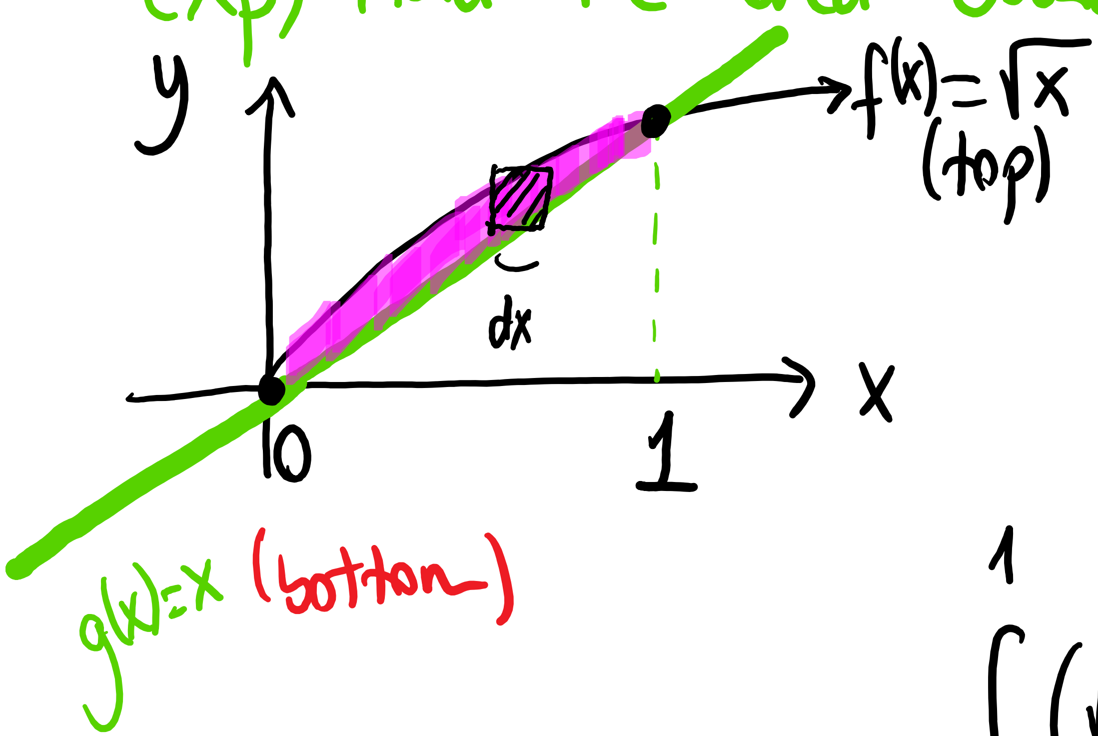
Area by using horizontal strips



$$A = \int_c^b (G(y) - F(y))\,dy + \int_b^d (F(y) - G(y))\,dy$$

yellow area purple area

Exp) Find the area bounded by the curve of $f(x) = \sqrt{x}$, $g(x) = x$



Use vertical strip / dx

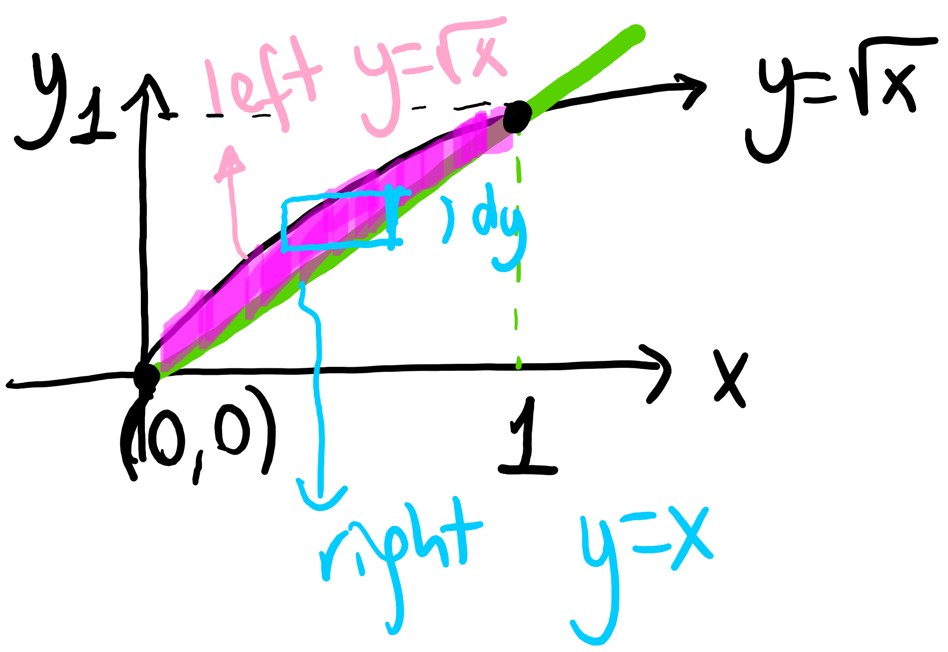
$\sqrt{x} = x \Rightarrow x = x^2 \Rightarrow x^2 - x = 0$

$x(x-1) = 0 \Rightarrow x = 0, 1$

$$\int_0^1 (\underbrace{\sqrt{x}}_{\text{top}} - \underbrace{x}_{\text{bottom}})\,dx = \left(\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right)\Bigg|_0^1 = \frac{1}{6}$$

$x = G(y)$

Exp) Find the area bounded by the curve of $y=\sqrt{x}$, $y=x$



$\sqrt{x} = x \Rightarrow x = x^2 \Rightarrow x^2 - x = 0$

$x(x-1) = 0 \Rightarrow x = 0, 1$ ✓

Use horizontal ~~strip~~ dy

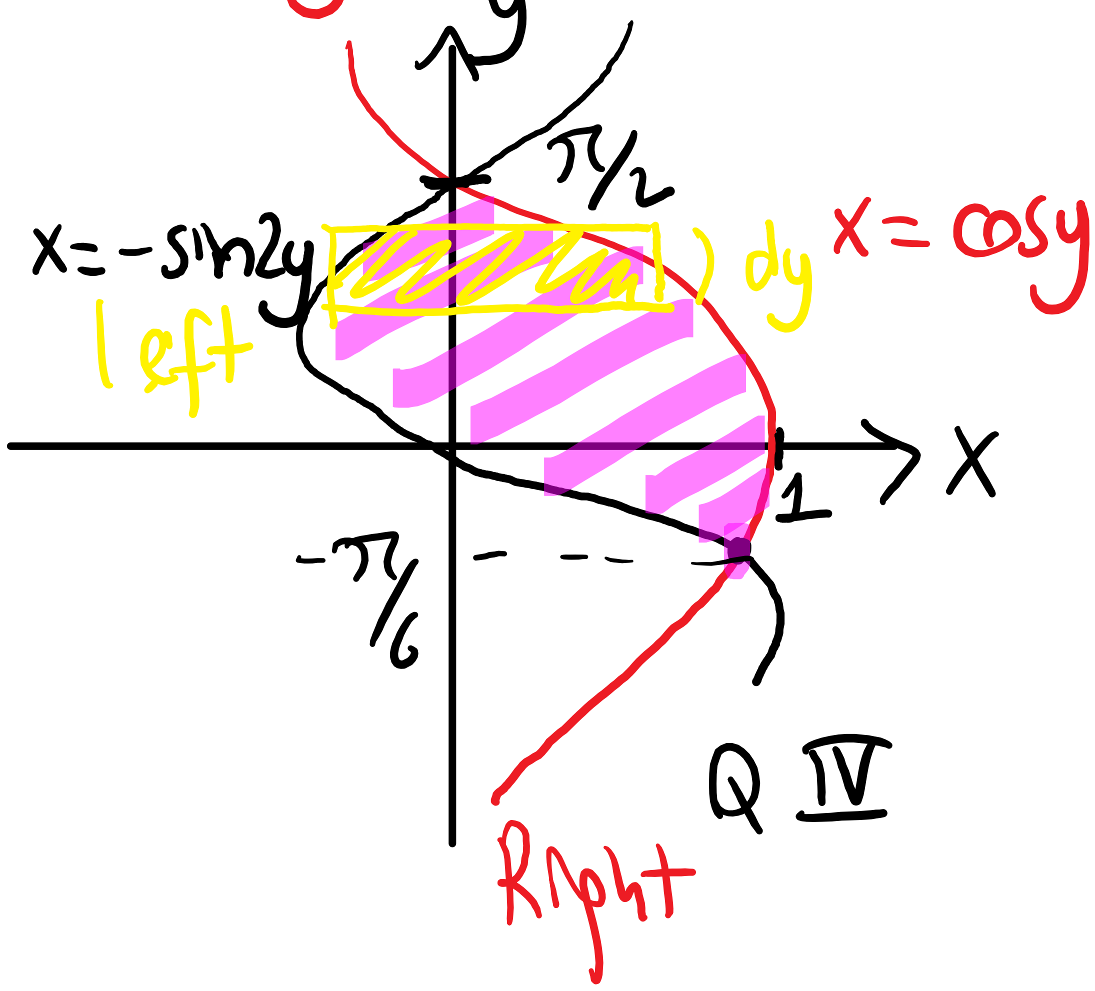
Re-write as $x = ?$

$y=\sqrt{x} \Rightarrow y^2 = x \Rightarrow x = y^2$ | $y = x \Rightarrow x = y$

$$\int_{y\to 0}^{y\to 1} (\underbrace{y}_{R} - \underbrace{y^2}_{L})\, dy = \left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^1 = \left(\frac{1}{2} - \frac{1}{3} - 0\right) = \frac{1}{6}$$

* We have to get the same area w/ horizontal & vertical strips.

You try it! Determine the area of the shaded region:



Hint: $\sin 2y = 2\cdot\sin y\cdot\cos y$

$(x, y) \to (\cos x,\ \sin y)$

$(+,\ -)$

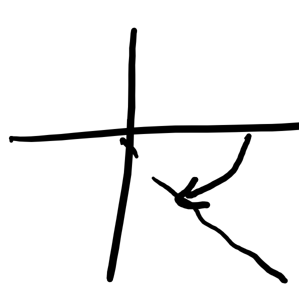
$y = ?$

$$\cos y = -\sin 2y = -2\cdot\sin y\cdot\cos y \Rightarrow \cos y + 2\cdot\sin y\cdot\cos y = 0$$

$$\underbrace{\cos y}_{0}\cdot\underbrace{(1+2\cdot\sin y)}_{0} = 0$$

$\cos y = 0 \Rightarrow y = \pi/2$ (given)

$1 + 2\cdot\sin y = 0 \Rightarrow \sin y = -1/2 \Rightarrow \arcsin y = \arcsin\left(-\frac{1}{2}\right)$



unit circle

$\frac{11\pi}{6} = \frac{-\pi}{6}$

$\sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2}$

since y is negative: $y = \frac{-\pi}{6}$

$$\boxed{\int_{\text{bottom } y}^{\text{top } y} (R-L)\,dy}$$

informally

$$\int_{-\pi/6}^{\pi/2} (\cos y - (-\sin 2y))\,dy = \int_{-\pi/6}^{\pi/2} (\cos y + \sin 2y)\,dy$$

$$= \left(\sin y - \frac{\cos 2y}{2}\right)\Big|_{-\pi/6}^{\pi/2} = \left(\sin\frac{\pi}{2} - \frac{\cos(2\cdot(\pi/2))}{2}\right) - \left(\sin\left(-\frac{\pi}{6}\right) - \frac{\cos\left(2\cdot -\frac{\pi}{6}\right)}{2}\right)$$

$$= \left(1 - \frac{(-1)}{2} - \left(-\frac{1}{2} - \frac{1/2}{2}\right)\right)$$

$$= \left(1 + \frac{1}{2} - \underbrace{\left(-\frac{1}{2} - \frac{1}{4}\right)}_{-3/4}\right) = \frac{3}{2} + \frac{3}{4} = \frac{9}{4}$$

(2)