8.8 Numerical Integration

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motivation:

Evaluate def. integral : $\int_a^b f(x)\,dx$ FTC

Exact Value

Approximate the area under the curve of $f(x)$ on $[a,b]$.

1) Riemann Sum (Rectangles) → ✓
2) Trapezoid Rule (Trapezoid)
3) Simpson Rule (parabola)

Approximation has errors (abs., relative)

If the integrand may not have an obvious antiderivative (e.g.: $\cos x^2$, $\frac{1}{\ln x}$) then we use Numerical methods to get accurate approximations.

Errors (that comes with approx.)

$x \to$ exact solution

$c \to$ computed numerical solution

Absolute error: $|c-x|$

Relative error: $\dfrac{|c-x|}{|x|}$ (if $x \neq 0$)

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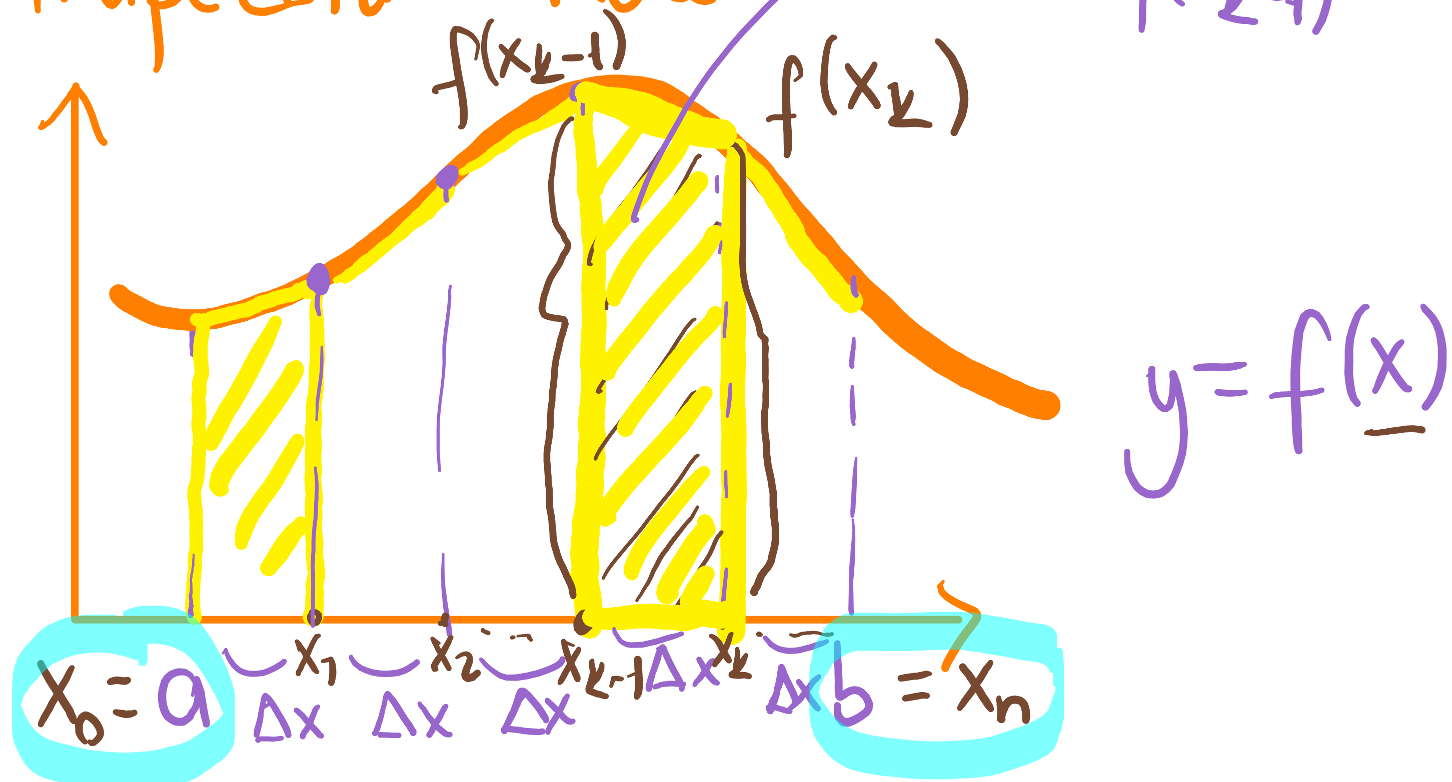
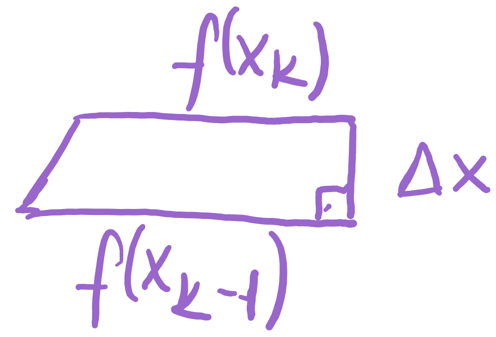
Trapezoid Rule

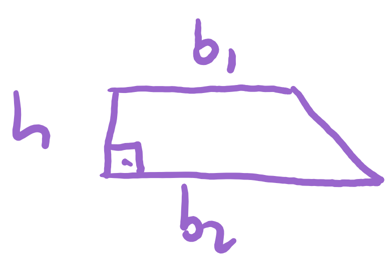




$$A_t = \frac{(f(x_k)+f(x_{k-1}))\cdot\Delta x}{2}$$

$$\underbrace{\Delta x = \frac{b-a}{n}}$$

Recall: Area of a Trapezoid



$$A = \frac{(b_1+b_2)\cdot h}{2}$$

$$\boxed{\underbrace{T(n)}_{\text{trapezoid rule approx to the integral}} = \left(\frac{1}{2}\cdot f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}\cdot f(x_n)\right)\cdot \Delta x}$$

b) $\int_2^4 x^2 dx = \frac{x^3}{3} \Big|_2^4 = \frac{4^3}{3} - \frac{2^3}{3} = \frac{64-8}{3} = \frac{56}{3}$

≈ 18.6 vs. $T(4) = 18.75$

c) Abs. Error $= |c - x| = \left| 18.75 - \underbrace{18.\overline{6}}_{\frac{56}{3}} \right| = 0.08\overline{3}$

Relative Error $= \frac{|c-x|}{|x|} = \frac{0.08\overline{3}}{56/3} = 0.00446$

$= 0.446\,\%$

* Do not round intermediate values

Exp) a) Approximate $\int_2^4 x^2\,dx$ by using the Trapezoid Rule with $\underline{n=4}$ subintervals

b) Find the exact area

$f(x)=x^2$

c) Find the absolute error and the relative error.

Given: $[a,b]=[2,4]$ $n=4$

$\boxed{x_k = x_{k-1} + \Delta x}$

$\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{2}{4} = \frac{1}{2}$

$x_0 = a = 2$ $f(2) = 2^2$

$x_1 = 2+0.5 = 2.5$ $f(2.5) = 6.25$

$x_2 = 2.5+0.5 = 3$ $f(3) = 3^2 = 9$

$x_3 = 3+0.5 = 3.5$ $f(3.5) = 12.25$

$x_4 = 3.5+0.5 = 4$ $f(4) = 4^2 = 16$

$$T(n) = \left(\frac{1}{2}\cdot f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2} f(x_n)\right)\cdot \Delta x$$

$$T(4) = \left(\frac{1}{2}\cdot f(2) + f(2.5) + f(3) + f(3.5) + \frac{1}{2}\cdot f(4)\right)\cdot \frac{1}{2}$$

$$= \left(\frac{1}{2}\cdot 4 + 6.25 + 9 + 12.25 + \frac{1}{2}\cdot 16\right)\cdot \frac{1}{2} = 18.75$$

You Try It!

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Find the Trapezoid Rule approximation to:

$$\int_0^1 \sin(\pi x)\,dx \qquad n=6 \qquad [0,1]$$

A) $(2-\sqrt{4})\cdot\frac{1}{3}$ B) $(2-\sqrt{3})\cdot\frac{1}{4}$ C) $(3-\sqrt{2})\cdot\frac{1}{2}$ D) $(2+\sqrt{3})\cdot\frac{1}{6}$

$$\boxed{T(n)=\left(\frac{1}{2}\cdot f(x_0)+\sum_{k=1}^{n-1} f(x_k)+\frac{1}{2}\cdot f(x_n)\right)\cdot\Delta x}$$

Given: $[a,b]\to[0,1]$ $\quad n=6 \quad \Delta x=\frac{1-0}{6}=\frac{1}{6}$

$x_0=0, \qquad f(0)=\sin(\pi\cdot 0)=0$

$x_1=0+\frac{1}{6} \qquad f\left(\frac{1}{6}\right)=\sin\left(\pi\cdot\frac{1}{6}\right)=\frac{1}{2}$

$x_2=\frac{1}{6}+\frac{1}{6} \qquad f\left(\frac{2}{6}\right)=\sin\left(\pi\cdot\frac{2}{6}\right)=\frac{\sqrt{3}}{2}$

$x_3=\frac{2}{6}+\frac{1}{6} \qquad f\left(\frac{3}{6}\right)=\sin\left(\pi\cdot\frac{3}{6}\right)=1$

$x_4=\frac{3}{6}+\frac{1}{6} \qquad f\left(\frac{4}{6}\right)=\sin\left(\pi\cdot\frac{4}{6}\right)=\frac{\sqrt{3}}{2}$

$x_5=\frac{4}{6}+\frac{1}{6} \qquad f\left(\frac{5}{6}\right)=\sin\left(\pi\cdot\frac{5}{6}\right)=\frac{1}{2}$

$x_6=\frac{5}{6}+\frac{1}{6}=1 \qquad f(1)=\sin(\pi\cdot 1)=0$

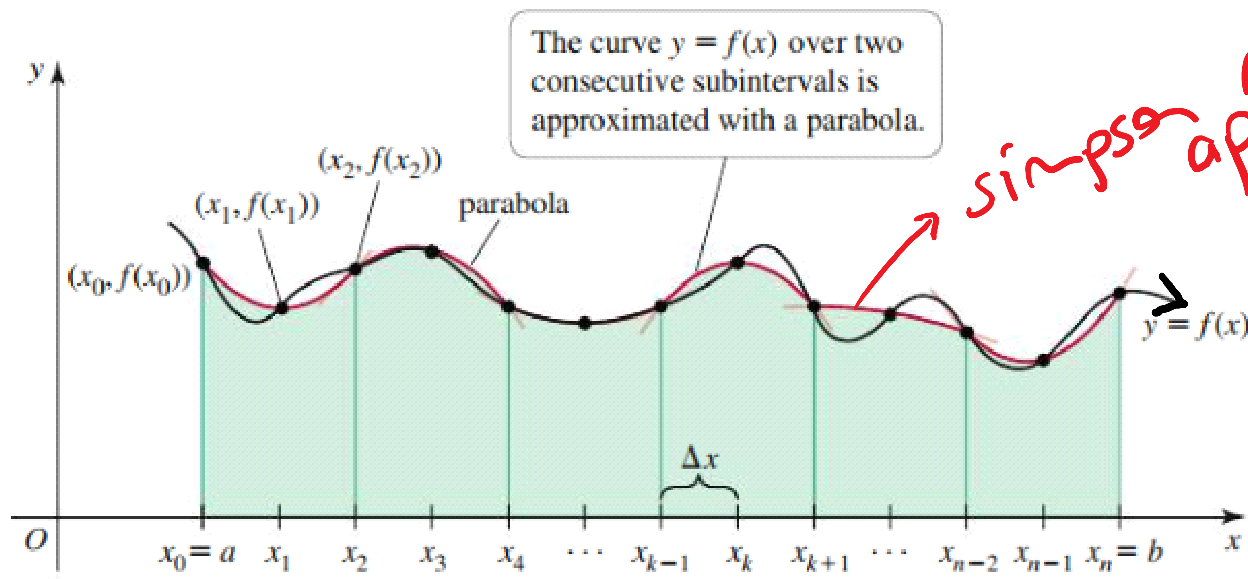
$$T(6)=\left(\cancel{\frac{1}{2}\cdot 0}+\frac{1}{2}+\frac{\sqrt{3}}{2}+1+\frac{\sqrt{3}}{2}+\frac{1}{2}+\cancel{\frac{1}{2}\cdot 0}\right)\cdot\frac{1}{6}$$

$$=\left(2+\sqrt{3}\right)\cdot\frac{1}{6} \qquad D$$

$\pi=180°$ Use the correct mode (radians) in your calculator.

Simpson Rule

Tuesday, January 26, 2021 8:03 AM



simpson rule approx.

We use parabolas to approx.

In each sub-interval pick 3 points on $f(x)$ and find the net area then apply this idea to every group of 3 consecutive points on $[a,b]$

Simpson's Rule: $\int_a^b f(x)\,dx$

$$\approx \left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)\frac{\Delta x}{3}$$

$$\boxed{S(n) = \left(f(x_0) + \underbrace{4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots}_{\text{pattern of } 4-2} + f(x_n)\right)\cdot\frac{\Delta x}{3}}$$

n must be an even integer

Exp) Approx. $\displaystyle\int_{\pi/2}^{5\pi/2} \frac{\sin(x)}{x}\,dx$ using Simpson's Rule, $n=4$.

$$\Delta x = \frac{b-a}{n} = \frac{\frac{5\pi}{2} - \frac{\pi}{2}}{4} = \frac{\pi}{2}$$

$$x_0 = \frac{\pi}{2},\quad x_1 = \frac{\pi}{2} + \frac{\pi}{2} = \pi,\quad x_2 = \pi + \frac{\pi}{2} = \frac{3\pi}{2},\quad x_3 = \frac{3\pi}{2} + \frac{\pi}{2} = 2\pi,\quad x_4 = \frac{5\pi}{2}$$

$$S(4) = \left(\frac{\cancelto{1}{\sin(\frac{\pi}{2})}}{\pi/2} + 4\cdot\frac{\cancelto{0}{\sin(\pi)}}{\pi} + 2\cdot\frac{\cancelto{-1}{\sin(\frac{3\pi}{2})}}{3\pi/2} + 4\cdot\frac{\cancelto{0}{\sin(2\pi)}}{2\pi} + \frac{\cancelto{1}{\sin(\frac{5\pi}{2})}}{\frac{5\pi}{2}}\right)\cdot\frac{\pi/2}{3}$$

$$= \left(f\left(\frac{\pi}{2}\right) + 4\cdot f(\pi) + 2\cdot f\left(\frac{3\pi}{2}\right) + 4\cdot f(2\pi) + f\left(\frac{5\pi}{2}\right)\right)\cdot\frac{\Delta x}{3}$$

$$= \left(\frac{1}{\pi/2} + 0 + \frac{(-2)}{3\pi/2} + 0 + \frac{1}{\frac{5\pi}{2}}\right)\cdot\frac{\pi}{6} = \left(\frac{2}{\cancel{\pi}} - \frac{4}{3\cancel{\pi}} + \frac{2}{5\cancel{\pi}}\right)\cdot\frac{\cancel{\pi}}{6}$$

$$\approx 0.178$$

POLL Q: Estimate the integral using the Trapezoidal Rule and Simpson's rule for the given value of n.

$f(8)=5$

$\to b\ (x_n)$

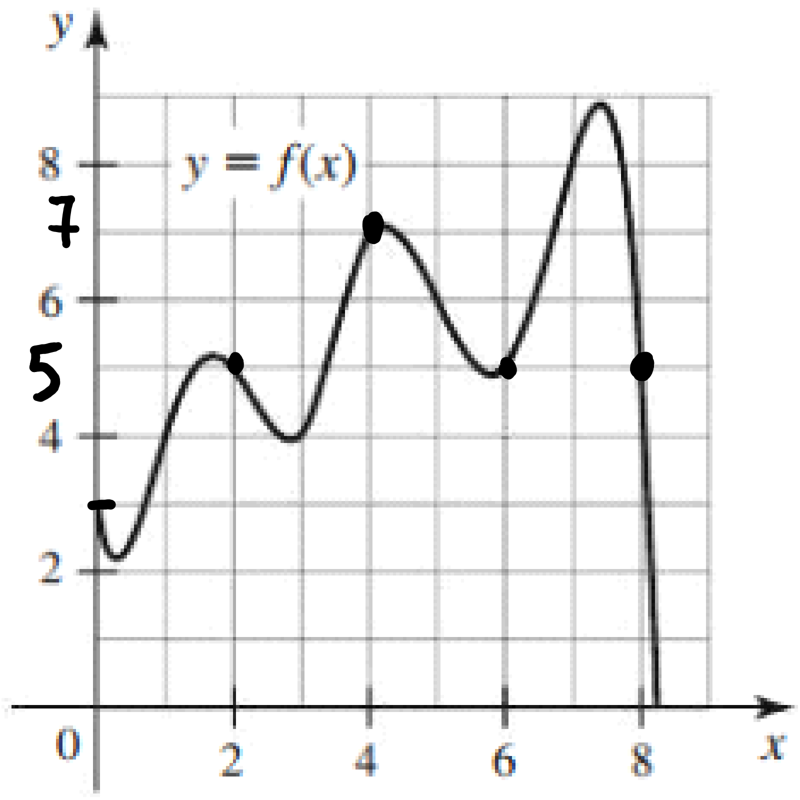
$\int_0^8 f(x)\,dx;\ n = 4$ (see figure)

$\hookrightarrow a\ (x_0)$

$\Delta x = \dfrac{b-a}{n} = \dfrac{8-0}{4} = 2$

$x_0 = a = 0,\ x_1 = 2,\ x_2 = 4,\ x_3 = 6,\ x_4 = b = 8$

$f(0)=3,\ f(2)=5,\ f(4)=7,\ f(6)=5,\ f(8)=5$



Trapezoid Rule w/ n=4

$T(4) = \left(\frac{1}{2}\cdot f(0) + f(2) + f(4) + f(6) + \frac{1}{2} f(8)\right)\cdot 2 = \left(\frac{1}{2}\cdot 3 + 5 + 7 + 5 + \frac{1}{2}\cdot 5\right)\cdot 2 = 42$ ✓

Simpson Rule w/ n=4

$\Delta x = 2$

$S(4) = \left(f(0) + \underbrace{4\cdot f(2) + 2\cdot f(4) + 4\cdot f(6)}_{\text{pattern of } 4-2} + f(8)\right)\cdot \frac{2}{3}$

$= \left(3 + 4\cdot 5 + 2\cdot 7 + 4\cdot 5 + 5\right)\cdot \frac{2}{3} = \frac{124}{3} \approx 41.\overline{3}$ ✓