EXAMPLE 5 **Derivatives of integrals** Use Part 1 of the Fundamental Theorem to simplify the following expressions.

a. $\frac{d}{dx}\int_1^x \sin^2 t\, dt$ **b.** $\frac{d}{dx}\int_x^5 \sqrt{t^2+1}\, dt$ **c.** $\frac{d}{dx}\int_0^{x^2} \cos t^2\, dt$

THEOREM 5.3 (PART 1) **Fundamental Theorem of Calculus**
If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\, dt, \quad \text{for} \quad a \le x \le b,$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\, dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.

c) $\frac{d}{dx}\int_0^{x^2} \cos t^2 \cdot dt$ $\left(\frac{dy}{dx} = ?\right)$

$t \to$ dummy var.

let $x^2 = u$

$2x = \frac{du}{dx}$

$2x \cdot dx = du$

$y = \int_0^u \cos t^2 \cdot dt$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\frac{dy}{dx} = \frac{d}{du}\left(\int_0^u \cos t^2 \cdot dt\right) \cdot 2x$

FTC 1

$= \cos u^2 \cdot 2x$

$= \cos (x^2)^2 \cdot 2x = 2x \cdot \cos x^4$

Warm Up - Evaluate the definite integral.

$$\int_0^{\pi} f(x)\,dx, \text{ where } f(x) = \begin{cases} \sin x + 1 & \text{if } x \le \pi/2 \\ 2\cos x + 2 & \text{if } x > \pi/2 \end{cases}$$

a) $2\pi + 1$
b) $1.5\pi - 2$
c) $1.5\pi - 1$
d) $2\pi - 1$

$$\int_0^{\pi/2} (\sin x + 1)\,dx + \int_{\pi/2}^{\pi} (2\cos x + 2)\,dx$$

$$= \left(-\cos x + x\right)\Big|_0^{\pi/2} + \left(2\cdot \sin x + 2x\right)\Big|_{\pi/2}^{\pi}$$

$(-\cos x)' = \sin x$

$(\sin x)' = \cos x$

$$= \left(-\cancel{\cos \frac{\pi}{2}}^{\,0} + \frac{\pi}{2}\right) - \left(-\cos 0 + 0\right) + \left(2\cancel{\sin \pi}^{\,0} + 2\pi\right) - \left(2\sin \frac{\pi}{2} + \frac{2\cdot\pi}{2}\right)$$

$$= \left(\frac{\pi}{2}\right) - \left(-1\right) + \left(0 + 2\pi\right) - \left(2 + \pi\right) = \frac{\pi}{2} + 1 + 2\pi - 2 - \pi$$

$$= \frac{\pi}{2} + \pi - 1 = 1.5\pi - 1$$

Objective:

Given just about any differentiable function, with enough know-how and persistence, you can compute its derivative. But the same cannot be said of antiderivatives. Many functions, even relatively simple ones, do not have antiderivatives that can be expressed in terms of familiar functions. Examples are $\sin x^2$, $(\sin x)/x$, and x^x. The immediate goal of this section is to enlarge the family of functions for which we can find antiderivatives.

Introductory Example: "u Substitution method", Backward Thinking to Derivatives, Inside/Outside Functions

$$\int (x^2+3x+5)\cdot(2x+3)\cdot dx$$

let $x^2+3x+5 = u$

$(2x+3)\cdot dx = du$

$$\int u\cdot du = \frac{u^2}{2} + C = \frac{(x^2+3x+5)^2}{2} + C$$

verify: $\left(\frac{(x^2+3x+5)^2}{2} + C\right)' = \frac{\cancel{2}\cdot(x^2+3x+5)^1\cdot(2x+3)}{\cancel{2}} + 0$

Table 5.6 General Integration Formulas

1. $\int \cos ax\, dx = \frac{1}{a}\sin ax + C$

2. $\int \sin ax\, dx = -\frac{1}{a}\cos ax + C$

3. $\int \sec^2 ax\, dx = \frac{1}{a}\tan ax + C$

4. $\int \csc^2 ax\, dx = -\frac{1}{a}\cot ax + C$

5. $\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax + C$

6. $\int \csc ax \cot ax = -\frac{1}{a}\csc ax + C$

7. $\int e^{ax}\, dx = \frac{1}{a}e^{ax} + C$

8. $\int b^x\, dx = \frac{1}{\ln b}b^x + C, b > 0, b \neq 1$

Chapter 5-5 Substitution Rule for Integrals

Sunday, December 6, 2020 8:02 PM

THEOREM 5.6 Substitution Rule for Indefinite Integrals
Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

PROCEDURE Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Disclaimer: Not all integrals yield to the Substitution Rule.

identify
inside/outside f.

EXAMPLE 1 Perfect substitutions Use the Substitution Rule to find the following indefinite integrals. Check your work by differentiating.

outside f.

a. $\int 2(2x+1)^3\,dx$ **b.** $\int 10e^{10x}\,dx$

inside f.

Let $(2x+1) = u$

$2 \cdot dx = du \quad \left(2 = \frac{du}{dx}\right)$

a. $\int 2\,(2x+1)^3 \cdot dx$

$= \int du \cdot u^3 = \int u^3 du = \frac{u^4}{4} + C = \frac{(2x+1)^4}{4} + C$

b. $\int 10 \cdot e^{10x}\,dx$

$u = 10x \Rightarrow du = 10 \cdot dx$

$\int e^u \cdot du = e^u + C = e^{10x} + C$

verify: $(e^{10x}+C)' = e^{10x} \cdot 10$ (integrand)

~~$x^4 = u$~~
~~$4x^3 \cdot dx = du$~~
?

EXAMPLE 2 **Introducing a constant** Find the following indefinite integrals.

a. $\int x^4(x^5+6)^9\,dx$ **b.** $\int \cos^3 x \sin x\,dx$

a. $\int x^4 (x^5+6)^9 \cdot dx$ → outside f.

inside f.

$u = x^5+6$

$\frac{du}{5} = \frac{5x^4 \cdot dx}{5}$

$\frac{du}{5} = x^4 \cdot dx$

$$= \int u^9 \cdot \frac{du}{5}$$

$$= \frac{1}{5} \cdot \frac{u^{10}}{10} + C = \frac{(x^5+6)^{10}}{50} + C$$

b. $\int \cos^3 x \cdot \sin x\,dx$ → outside f.

inside f.

~~$\cos^3 x = u$~~
~~$3 \cdot \cos^2 x \cdot (-\sin x) \cdot dx = du$~~

multiply by (-1)

$u = \cos x \Rightarrow du = (-\sin x) \cdot dx \Rightarrow -du = \sin x \cdot dx$ ✓

$$\int u^3 \cdot (-du) = -\int u^3 \cdot du = -\frac{u^4}{4} + C = -\frac{(\cos x)^4}{4} + C$$

Definite Integrals

The Substitution Rule is also used for definite integrals; in fact, there are two ways to proceed.

- You may use the Substitution Rule to find an antiderivative F and then use the Fundamental Theorem to evaluate $F(b) - F(a)$.
- Alternatively, once you have changed variables from x to u, you also may change the limits of integration and complete the integration with respect to u. Specifically, if $u = g(x)$, the lower limit $x = a$ is replaced with $u = g(a)$ and the upper limit $x = b$ is replaced with $u = g(b)$.

Option 1: Put it in terms of x, keep the intervals of integration wrt x

Option 2: Keep u, change the intervals of integration wrt u

The second option tends to be more efficient, and we use it whenever possible. This approach is summarized in the following theorem, which is then applied to several definite integrals.

THEOREM 5.7 Substitution Rule for Definite Integrals
Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du. \qquad (2)$$

Exp) Evaluate $\int_1^2 (4x-5)^3 \cdot dx$ **Use Option 1.** (keep x)

outside f.

inside f.

$$4x-5=u \Rightarrow \frac{4\cdot dx = du}{4}, \quad dx = \frac{du}{4}$$

temp. use indef. integral

$$\int u^3 \cdot \frac{du}{4} = \frac{u^4}{4\cdot 4} + C = \frac{u^4}{16} + C = \frac{(4x-5)^4}{16} + C$$

$$\left(\frac{(4x-5)^4}{16} + C\right)\Bigg|_1^2 = \frac{(4\cdot 2-5)^4}{16} - \frac{(4\cdot 1-5)^4}{16} = \frac{81}{16} - \frac{1}{16} = 5$$

FTC 2

Exp) Evaluate $\int_1^2 (4x-5)^3 \cdot dx$ Use Option 2. (keep u)

$\underbrace{4x-5=u}$ $\Rightarrow$ $4 \cdot dx = du$ $\Rightarrow$ $dx = \dfrac{du}{4}$

Change limits of integration:

$x=2 \Rightarrow u = 4x-5 = 4\cdot 2 - 5 = 3$

$x=1 \Rightarrow u = 4x-5 = 4\cdot 1 - 5 = -1$

$$\int_{-1}^{3} u^3 \cdot \frac{du}{4} = \frac{u^4}{4\cdot 4}\Bigg|_{-1}^{3} = \frac{3^4}{16} - \frac{(-1)^4}{16} = \frac{81-1}{16} = 5$$

Sunday, December 6, 2020 8:30 PM

Option 1: keep x

EXAMPLE 5 **Definite integrals** Evaluate the following integrals.

b. $\int_2^3 \frac{x^2}{x^3 - 7}\,dx$

c. $\int_0^{\pi/2} \underbrace{\sin^{4} x}_{\text{inside f.}} \cos x\,dx$ → outside f.

$$= \int_0^{\pi/2} (\sin x)^4 \cdot \cos x \cdot dx$$

$$u = \sin x \Rightarrow du = \cos x \cdot dx$$

$$\int u^4 \cdot du = \frac{u^5}{5} + c \quad \Big\} \left(\frac{(\sin x)^5}{5} + C\right)\Bigg|_0^{\pi/2}$$

$$= \frac{(\sin \pi/2)^5}{5} - \frac{(\sin 0)^5}{5} = \frac{1}{5} - 0 = \frac{1}{5}$$

You try it! Evaluate the definite integral.

$$\int_0^1 \frac{v^3+1}{\underbrace{\sqrt{v^4+4v+4}}_{u}}\,dv$$

Option 2
Keep u

$$u = v^4+4v+4$$

$$du = (4v^3+4)\cdot dv$$

$$\frac{du}{4} = \frac{4(v^3+1)\cdot dv}{4}$$

$$\frac{du}{4} = (v^3+1)\cdot dv$$

$$v=1 \Rightarrow u = v^4+4v+4 = 1+4+4 = 9$$

$$v=0 \Rightarrow u = 0+0+4 = 4$$

$$\int_4^9 \frac{du/4}{\sqrt{u}} = \int_4^9 \frac{1}{4}\cdot u^{-1/2}\cdot du = \frac{1}{\cancel{4}_{2}}\cdot \frac{u^{1/2}}{\cancel{1/2}}\Bigg|_4^9$$

$$= \frac{1}{2}\sqrt{u}\,\Bigg|_4^9 = \frac{1}{2}\cdot\sqrt{9} - \frac{1}{2}\cdot\sqrt{4} = \frac{3}{2} - \frac{2}{2} = \frac{1}{2}$$

You try it! Dr. T's Fall 2019 Recitation Quiz Question.

Find the area of the region under the the graph of $y = \frac{e^{\sqrt{t}}}{\sqrt{t}}$ above the $t-$axis on the interval of $[4, 25]$. **Simplify your final answer as much as possible.**

A) $3(e^5-e^3)$ B) $2(e^5-e^2)$ C) $2(e^5-e^4)$ D) None

$$\int_4^{25} \frac{e^{\sqrt{t}}}{\sqrt{t}} \cdot dt$$

$$u = \sqrt{t} = t^{1/2}$$

$$du = \frac{1}{2} \cdot t^{-1/2} \cdot dt = \frac{dt}{2\sqrt{t}}$$

$$2 \cdot du = \frac{1}{2} \cdot \frac{dt}{\sqrt{t}} \cdot 2$$

$$2 \cdot du = \frac{dt}{\sqrt{t}} \quad ✓$$

$t = 25 \Rightarrow u = \sqrt{t} \Rightarrow u = \sqrt{25} = 5$

$t = 4 \Rightarrow u = \sqrt{t} \Rightarrow u = \sqrt{4} = 2$

$$\int_2^5 e^u \cdot 2 \cdot du = 2e^u \Big|_2^5 = 2e^5 - 2 \cdot e^2$$

$$= 2(e^5 - e^2)$$