1. Find all solutions to the following equation. If there is no solution, write "No Solution":

$$\ln\left(\frac{2x^2}{3-x}\right) - \ln(x) = \ln\left(\frac{4x}{4-x}\right)$$

Solution: By using the properties of logarithms obtain:

$$\ln\left(\frac{\frac{2x^2}{3-x}}{x}\right) = \ln\left(\frac{4x}{4-x}\right)$$

simplify the left-hand side of the equation as:

$$\ln\left(\frac{2x}{3-x}\right) = \ln\left(\frac{4x}{4-x}\right)$$

this implied that:

$$\left(\frac{2x}{3-x}\right) = \left(\frac{4x}{4-x}\right)$$

solve the rational equation by cross multiplying to obtain:

$$2x(4-x) = 4x(3-x)$$

the solution is $x = 2$ since it is in the domain of the original logarithm equation.

2. Solve the inequality $x^2 + 4x - 32 < 0$. Write your answer using interval notation.

Solution:

We use the cut-point (or sign chart) method. For our sign chart, the cut points are found by factoring the trinomial and finding the solutions(or cutpoints) by setting the trinomial to 0. Let $f(x) = x^2 + 4x - 32$. Therefore, we factor $f(x)$ as $(x+8)(x-4))$ and find the cut points $x = 4, x = -8$. Now we test the truth of the inequality using a test point from each corresponding sub-interval.

We can pick test points in each sub-interval to test the sign of the polynomial. Hint: use the factored form of the polynomial, $((x+8)(x-4))$, to make the calculation easier. Let test points be $x = -10, 0, 6$.

$$f(-10) = (-10+8)\cdot(-10-4) > 0$$

$$f(0) = (0+8)\cdot(0-4) < 0$$

$$f(6) = (6+8)\cdot(6-4) > 0$$

The sign chart is provided below:

$$f(x) = (x+8)(x-4)$$

$f(x)$ + + + + + + + 0 − − − − − − − 0 + + + + + + +

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-8 4

Therefore, the domain of $f(x) < 0$ in interval notation is: $(-8, -4)$.

3. Evaluate the limit or determine that it does not exist. If the limit does not exist, write "DNE". $\lim_{x\to 6}\left(\frac{36-x^2}{x-6}\right)$.

Solution: First factor the numerator, then simplify and use the direct substitution property to evaluate the limit:

$$\lim_{x\to 6}\left(\frac{36-x^2}{x-6}\right) = \lim_{x\to 6}\left(\frac{(6-x)(6+x)}{x-6}\right) = \lim_{x\to 6}\left(-(6+x)\right) = -12$$

4. Evaluate the limit or determine that it does not exist. If the limit does not exist, write "DNE". $\lim_{x\to 0}\left(\frac{2x^2}{\sin^2(5x)}\right)$.

Solution: Use the special limit $\lim_{\theta\to 0}\left(\frac{\sin(a\theta)}{a\theta}\right) = 1$ and some algebra.

$$\lim_{x\to 0}\left(\frac{2x^2}{\sin^2(5x)}\right) = \lim_{x\to 0}\left(\frac{2x}{\sin(5x)}\cdot\frac{x}{\sin(5x)}\right) = \lim_{x\to 0}\left(\frac{5x}{\sin(5x)}\cdot\frac{5x}{\sin(5x)}\cdot\frac{2}{5\cdot 5}\right) = \frac{2}{25}$$

5. Find the value of k that makes f continuous at $x = 1$ or determine that no such value of k exists. If there is no k value exists, write "DNE".

$$f(x) = \begin{cases} kx^3 + e^{x-1} & , \quad x < 1 \\ 3x - \ln(2x-1) & , \quad x \geq 1 \end{cases}$$

Solution: First we calculate the left-limit, right-limit, and function value at $x = 1$. In order for a function to be continuous at $x = 1$ all these values must be equal. Therefore, we have:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (kx^3 + e^{x-1}) = k + e^0 = k + 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3x - \ln(2x-1)) = 3 - \ln(1) = 3$$

$$f(1) = 3 - \ln(1) = 3$$

In order for $f(x)$ to be continuous at $x = 1$:

$$k + 1 = 3$$
$$k = 2$$

6. Calculate $g'(x)$. After calculating the derivative, do not simplify your answer.

$$g(x) = \ln(4x - \tan x)$$

Solution: Find the derivative of $g(x)$ by using the chain rule as:

$$g'(x) = (4x - \tan x)' \cdot \frac{1}{4x - \tan x} = \frac{4 - \sec^2(x)}{4x - \tan x}$$

7. Calculate $g'(x)$. After calculating the derivative, do not simplify your answer.

$$g(x) = (x - \sqrt{2x+3})^{1/3}$$

Solution: Find the derivative of $g(x)$ by using the extended power rule.

$$\begin{aligned} g'(x) &= \frac{1}{3} \cdot (x - \sqrt{2x+3})^{-2/3} \cdot (x - \sqrt{2x+3})' \\ &= \frac{1}{3} \cdot (x - \sqrt{2x+3})^{-2/3} \cdot \left(1 - \frac{1}{2}(2x+3)^{-1/2} \cdot 2\right) \end{aligned}$$

8. Find the coordinates of the points on the graph of $y = 2x^3 - 24x$ where the tangent line is horizontal. Your answer should be a list of ordered pairs.

Solution:

A horizontal line has a slope of 0 and the slope of the tangent line is given by the derivative. Therefore, we must solve the equation $y' = 0$.

$$y' = 6x^2 - 24 = 6(x^2 - 4) = 6(x-2)(x+2) \Longrightarrow x = 2, -2$$

Therefore, the points on the graph with a horizontal tangent is $(2, -32), (-2, 32)$.

For problems #9 – #14, you must show all work, and your work will be graded. Your explanation should be clear and coherent. You must use proper calculus methods and notation to receive full credit.

9. Evaluate the limit or determine that it does not exist. If the limit does not exist, write "DNE". You must use proper calculus methods and notation to receive full credit.

$$\lim_{x\to 1} f(x) \text{ , where } f(x) = \begin{cases} \dfrac{\sqrt{x}-1}{x-1} & , \quad x > 1 \\ 8 & , \quad x = 1 \\ \dfrac{2x-2}{x^2+2x-3} & , \quad x < 1 \end{cases}$$

Solution:

First we calculate the left-limit and right-limit to find out if they are the same and the two-sided limit exists at $x = 1$.

$$\lim_{x\to 1^+} \left(\frac{\sqrt{x}-1}{x-1}\right) = \lim_{x\to 1^+} \left(\frac{\sqrt{x}-1}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1}\right) = \lim_{x\to 1^+} \left(\frac{x-1}{(x-1)\cdot(\sqrt{x}+1)}\right) = \lim_{x\to 1^+} \left(\frac{1}{\sqrt{x}+1}\right) = \frac{1}{2}$$

$$\lim_{x\to 1^-} \left(\frac{2x-2}{x^2+2x-3}\right) = \lim_{x\to 1^-} \left(\frac{2(x-1)}{(x+3)(x-1)}\right) = \lim_{x\to 1^-} \left(\frac{2}{x+3}\right) = \frac{2}{4} = \frac{1}{2}$$

Since left-limit and right-limit are equal, the limit does exist and we write it as $\lim_{x\to 1} f(x) = \frac{1}{2}$

10. Find the values of a and b that make f continuous for all x or determine that no such values of a and b exist. You must use proper calculus methods and notation to receive full credit.

$$f(x) = \begin{cases} e^x - 2 & , \quad x \le 0 \\ \dfrac{b(x^2-1)}{1-x} & , \quad 0 < x < 1 \\ x^2 - a & , \quad x \ge 1 \end{cases}$$

Solution: Note that although the second piece of $f(x)$ is not continuous at $x = 1$, this piece of the function does not include $x = 1$ in its domain. We calculate the left-limit, right-limit, and function value at the transition points $x = 0, x = 1$.

$\lim_{x\to 0^-} (e^x - 2) = e^0 - 2 = 1 - 2 = -1$

$\lim_{x\to 0^+} \left(\frac{b(x^2-1)}{1-x}\right) = \lim_{x\to 0^+} \left(\frac{b(x-1)(x+1)}{1-x}\right) = \lim_{x\to 0^+} (-b(x+1)) = -b$

$f(0) = -1$

To make $f(x)$ continuous at $x = 0$; the left-limit, right-limit, and function value at $x = 0$ must all be equal. Therefore, we must have $-b = -1 \implies b = 1$. Then we calculate the left-limit, right-limit, and function value at $x = 1$:

$\lim_{x\to 1^+} (x^2 - 1) = 1 - a$

$\lim_{x\to 1^-} \left(\frac{b(x^2-1)}{1-x}\right) = \lim_{x\to 1^-} \left(\frac{b(x-1)(x+1)}{1-x}\right) = \lim_{x\to 1^-} (-b(x+1)) = -2b$

$f(1) = 1 - a$

To make $f(x)$ continuous at $x = 1$; the left-limit, right-limit, and function value at $x = 1$ must all be equal. Recall that $b = 1$. Therefore, we must have $1 - a = -2b \implies 1 - a = -2 \implies a = 3$.

11. You must use the limit definition of derivative and proper notation to receive full credit. If you simply quote a derivative rule without using the limit definition, you will receive no credit.

Let $f(x) = \dfrac{x-2}{x+1}$. Calculate $f'(-2)$.

Solution: Observe that $f(-2) = 4$.

$$\begin{aligned} f'(-2) &= \lim_{x\to -2} \frac{f(x) - f(-2)}{x - (-2)} \\ &= \lim_{x\to -2} \frac{\frac{x-2}{x+1} - 4}{x+2} \\ &= \lim_{x\to -2} \frac{\frac{x-2-4(x+1)}{x+1}}{x+2} \\ &= \lim_{x\to -2} \frac{\frac{-3x-6}{x+1}}{x+2} \\ &= \lim_{x\to -2} \frac{\frac{-3(x+2)}{x+1}}{x+2} \\ &= \lim_{x\to -2} \frac{-3}{x+1} \\ &= 3 \end{aligned}$$

12. Calculate $f'(x)$. After calculating the derivative, do not simplify your answer.

(a) $f(x) = \sqrt{x}(\sqrt{x} - x^{3/2}) + \pi^{1/3}$

Solution: First simplify then calculate the derivative:

$$f(x) = \sqrt{x}(\sqrt{x} - x^{3/2}) + \pi^{1/3} = x - x^2 + \pi^{1/3}$$

$$f'(x) = (x - x^2 + \pi^{1/3})' = 1 - 2x$$

(b) $f(x) = \dfrac{\sin(2x-4)}{x^2-4}$

Solution: Use the quotient rule and chain rule:

$$f'(x) = \frac{\cos(2x-4) \cdot 2(x^2-4) - \sin(2x-4) \cdot 2x}{(x^2-4)^2}$$

13. Find the x-coordinate of each point on the graph of $y = \frac{4+2x}{1-3x}$ where the tangent line is perpendicular to the line $y = \frac{-1}{14}x - 4$.

Solution: Since the tangent line to the graph of y at an unknown point is perpendicular to $y = \frac{-1}{14}x - 4$, the slope of the tangent line is negative reciprocal of the slope of the line $y = \frac{-1}{14}x - 4$. Therefore, the slope of the tangent line to this unknown point is 14. In order to find the x-coordinate of this point, find the derivative of $y = \frac{4+2x}{1-3x}$ by using the quotient rule:

$$y' = \frac{(4+2x)' \cdot (1-3x) - (4+2x) \cdot (1-3x)'}{(1-3x)^2} = \frac{2 \cdot (1-3x) - (4+2x) \cdot (-3)}{(1-3x)^2} = \frac{14}{(1-3x)^2}$$

Then, set the derivative to 14 to solve it for x:

$$\frac{14}{(1-3x)^2} = 14 \Longrightarrow (1-3x)^2 = 1 \Longrightarrow 1 - 3x = \pm 1 \Longrightarrow x = 0, \frac{2}{3}.$$

14. For both parts of this problem let $f(x) = x^2 - 4x + 1$.

(a) (2 points) Calculate $f'(3)$ by using derivative rules to receive full credit.

Solution:
Find the derivative of $f(x)$ by using the rules of derivatives: $f'(x) = 2x - 4$.
Evaluate the derivative at $x = 3$: $f'(3) = 2$

(b) (8 points) Calculate $f'(3)$ by using the limit definition of derivative and proper notation to receive full credit. If you simply quote a derivative rule without using the limit definition, you will receive no credit.

Solution: Observe that $f(3) = 3^2 - 4 \cdot 3 + 1 = -2$

$$\begin{aligned} f'(3) &= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \\ &= \lim_{x \to 3} \frac{(x^2 - 4x + 1) - (-2)}{x - 3} \\ &= \lim_{x \to 3} \frac{x^2 - 4x + 3}{x - 3} \\ &= \lim_{x \to 3} \frac{(x-3)(x-1)}{x - 3} \\ &= \lim_{x \to 3} (x - 1) \\ &= 2 \end{aligned}$$

As seen above, we should obtain the same value for the slope of the tangent line at a point by using both methods.