Chapter 3: Derivatives

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Derivatives describe change.

Almost everything around us is in a state of change!

$f \to$ quantity of interest

$\boxed{\text{derivative of } f \text{ is } f'}$

f' "f prime"

$f' \to$ slope of the curve $y=f(x)$ as it changes with respect to x. (wrt)

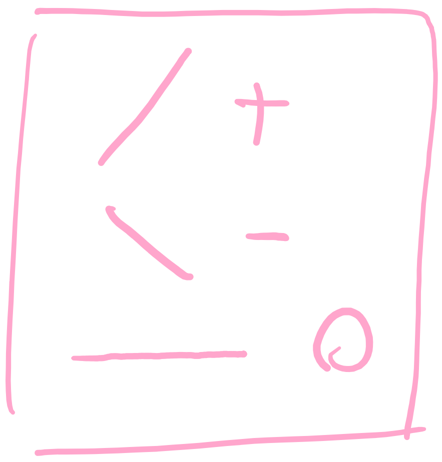
$f' \to$ instantaneous rate of change of f wrt x

3.1 Introducing the Derivative

In this section, we return to the problem of finding the slope of a line tangent to a curve, introduced at the beginning of Chapter 2. This problem is important for several reasons.

- We identify the slope of the tangent line with the *instantaneous rate of change* of a function (Figure 3.1).
- The slopes of the tangent lines as they change along a curve are the values of a new function called the *derivative*.
- Looking farther ahead, if a curve represents the trajectory of a moving object, the tangent line at a point on the curve indicates the direction of motion at that point (Figure 3.2).

Recall:



signs of slopes

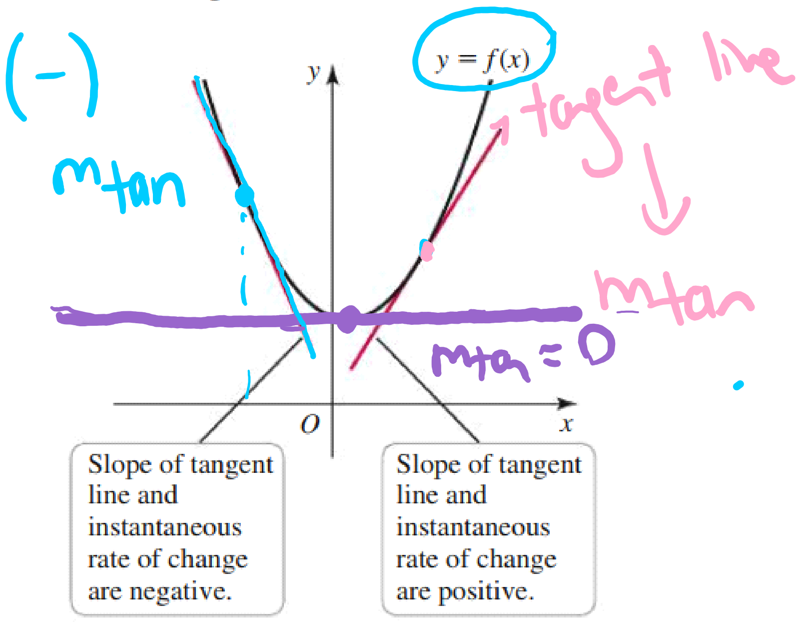


Figure 3.1

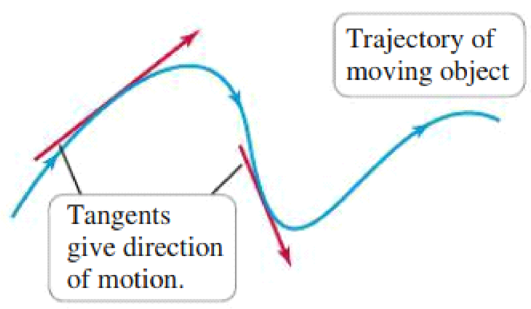


Figure 3.2

Recall: Instantaneous vs. Average Velocity

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($V_{inst.}$ vs. V_{ave})

slope of a tangent line (1 pt)   Slope of a secant line (2 points)

EXAMPLE 1 **Instantaneous velocity and tangent lines** A rock is launched vertically upward from the ground with an initial speed of 96 ft/s. The position of the rock in feet above the ground after t seconds is given by the function $s(t) = -16t^2 + 96t$. Consider the point $P(1, 80)$ on the curve $y = s(t)$.

a. Find the instantaneous velocity of the rock 1 second after launch and find the slope of the line tangent to the graph of s at P.

b. Find an equation of the tangent line in part (a).

Recall: $V_{ave} = \dfrac{s(t) - s(a)}{t - a}$ $[a, t]$

magical point $\to a = 1$ sec

$$V_{inst} = \lim_{t \to 1} \left(\frac{s(t) - s(a)}{t - a} \right)$$

$s(t) = -16t^2 + 96t$

$s(1) = -16 + 96 = 80$ ft.

Given: initial speed of 96 ft/s   "units"

$s(t) = -16t^2 + 96t$

$P(1, 80)$

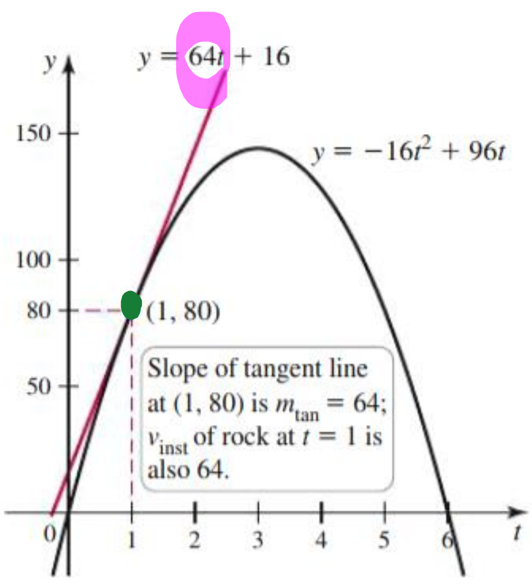
$P(t, s(t))$

$$V_{inst} = \lim_{t \to 1} \left(\frac{-16t^2 + 96t - 80}{t - 1} \right)$$

$$= \lim_{t \to 1} \left(\frac{-16(t-5)\cancel{(t-1)}}{\cancel{t-1}} \right) = \lim_{t \to 1} \left(-16(t-5)\right)$$

$$\overset{DSP}{=} -16(1-5)$$

$$V_{inst} = 64 \frac{ft}{sec}$$



$P(1, 80)$
(x_1, y_1)
$(t, s(t))$

b) $m_{tan} = 64$ ft/sec.

eq. of the tangent line to s(t) at P?

use point-slope form to re-write:

$$y - 80 = 64(t - 1)$$

$$y = 64t - 64 + 80$$

$$y = 64t + 16$$

Interpret: The instantaneous velocity of the rock at $t = 1$ is 64 ft/sec.

in other words:

The slope of the tangent line to the graph of s(t) at P(1, 80) is 64 ft/sec.

DEFINITION Rate of Change and the Slope of the Tangent Line

The **average rate of change** in f on the interval $[a, x]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

The **instantaneous rate of change** in f at a is

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \qquad (1)$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists. The **tangent line** is the unique line through $(a, f(a))$ with slope $m_{\tan}$. Its equation is

$$y - f(a) = m_{\tan}(x - a).$$

$$m_{\tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(if no point given)

$$m_{\tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

Familiar?

$h = ?$

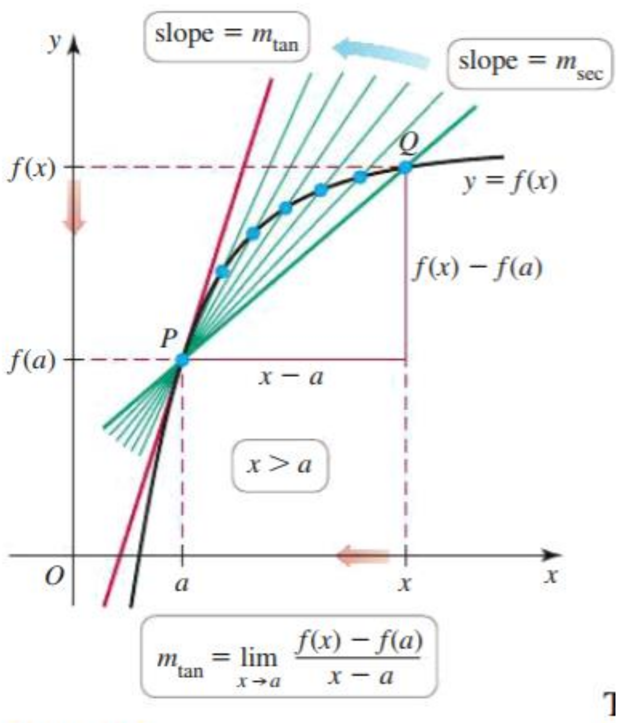


Figure 3.5

Check Interactive 3.5

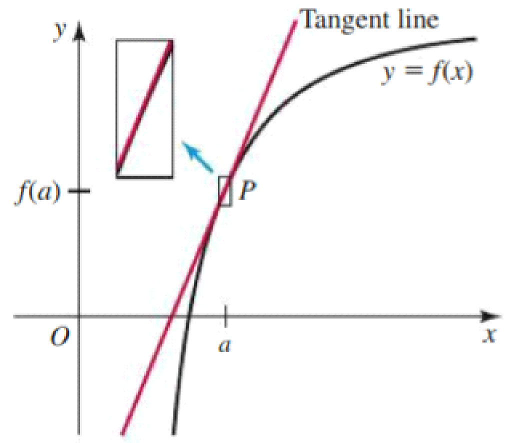


Figure 3.6

➤ The definition of m_{sec} involves a *difference quotient*, introduced in Section 1.1.

➤ If x and y have physical units, then the average and instantaneous rates of change have units of (units of y)/(units of x). For example, if y has units of meters and x has units of seconds, the units of the rates of change are meters/second (m/s).

ALTERNATIVE DEFINITION Rate of Change and the Slope of the Tangent Line

The **average rate of change** in f on the interval $[a, a + h]$ is the slope of the corresponding secant line:

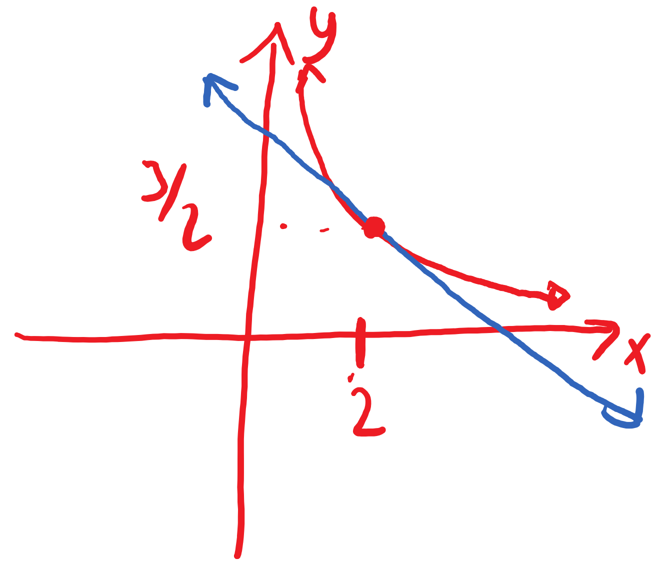
$$m_{\text{sec}} = \frac{f(a + h) - f(a)}{h}.$$

The **instantaneous rate of change** in f at a is

$$m_{\tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}, \qquad (2)$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists.

EXAMPLE 2 **Equation of a tangent line** Find an equation of the line tangent to the graph of $f(x) = \frac{3}{x}$ at $\left(2, \frac{3}{2}\right)$.



$$m_{tan} = \lim_{x \to 2} \left(\frac{f(x) - f(2)}{x-2} \right)$$

Given:

$$f(x) = \frac{3}{x}$$

$$f(2) = \frac{3}{2}$$

$$= \lim_{x \to 2} \left(\frac{\frac{3}{x}(2) - \frac{3}{2}(x)}{x-2} \right)$$

$$= \lim_{x \to 2} \left(\frac{\frac{6-3x}{2x}}{x-2} \right)$$

$$-3(-2+x)$$

$$-2+x = x-2$$

$$= \lim_{x \to 2} \left(\frac{6-3x}{2x} \cdot \frac{1}{x-2} \right)$$

$$= \lim_{x \to 2} \left(\frac{-3\cancel{(-2+x)}}{2x} \cdot \frac{1}{\cancel{x-2}} \right)$$

$$= \lim_{x \to 2} \left(\frac{-3}{2x} \right) \overset{DSP}{=} \boxed{\frac{-3}{4} = m_{tan}}$$

eq: $m_{tan} = \frac{-3}{4}$, $P\left(2, \frac{3}{2}\right)$

we point-slope form

$$y - \frac{3}{2} = \frac{-3}{4}(x-2) \Rightarrow y = \frac{-3}{4}x + 3$$